THEOREM OF THE DAY

Perfect's Necklace Formula *Take n balls of t different colours, with n_i balls of colour i, $i = 1, \ldots, t$. Let $1 = d_1 < \ldots < d_m = \gcd(n_1, \ldots, n_t)$ be the sequence of divisors common to all the n_i. Then the number of distinct arrangements of the balls, in a line or in a circle, is given, respectively, by:*

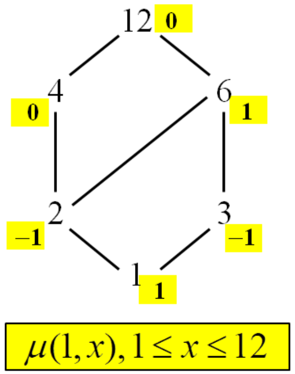
Linear: $\displaystyle \binom{n}{n_1, \ldots, n_t} = \frac{n!}{n_1! \cdots n_t!}$; **Circular:** $\displaystyle \frac{1}{n}\sum_{i=1}^{m}\sum_{j=i}^{m} d_i \binom{n/d_j}{n_1/d_j, \ldots, n_t/d_j}\mu(d_i, d_j)$, *where $\mu(x, y)$ is the Möbius function on the partially ordered set of divisors.*

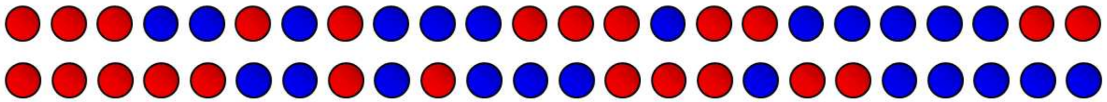
The Möbius Function
If $\leq$ is a partial order then the Möbius function is defined recursively as:

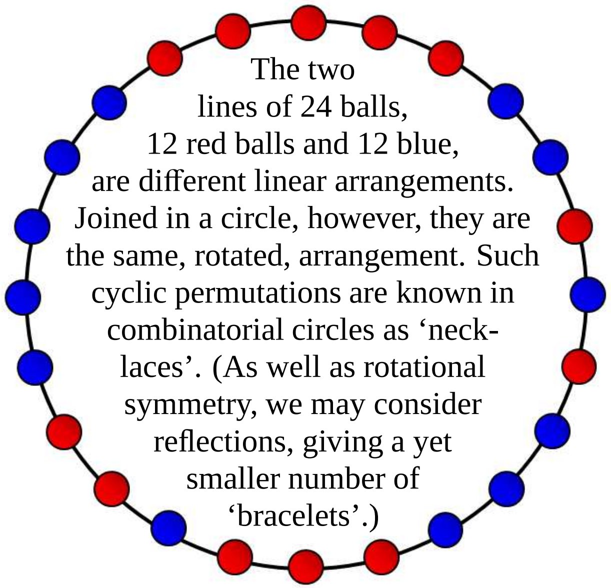
$$\mu(x,y) = \begin{cases} 1 & \text{if } x = y \\ -\sum_{x \leq z < y} \mu(x,z) & \text{otherwise} \end{cases}$$

(the summation evaluating to zero if $x \nleq y$).

The partial order on the right shows the positive divisors of 12, with lines showing the ordering by 'immediate divisibility'. The values of $\mu(x, y)$ have been calculated for $x = 1$, beginning with $\mu(1, 1) = 1$ and working up the diagram.







In our example there are $n = 24$ balls with $n_1 = n_2 = 12$. In this case, with $t = 2$ colours, the multinomial coefficient $\binom{24}{12,12}$ is just the usual binomial coefficient and evaluates to 2704156.
For the necklace count we list the divisors of $\gcd(12, 12) = 12$ as $d_1 = 1, d_2 = 2, d_3 = 3, d_4 = 4, d_5 = 6$, and $d_6 = 12$.
The Möbius function values may be put in an array as shown on the right (all entries below the diagonal are zero). And now we can calculate the double sum. For example, at $i = 3$ the inner sum is:

	1	2	3	4	6	12
1	1	−1	−1	0	1	0
2		1	0	−1	−1	1
3			1	0	−1	0
4				1	0	−1
6					1	−1
12						1

$$3 \times \left(\binom{8}{4} \times 1 + \binom{6}{3} \times 0 + \binom{4}{2} \times -1 + \binom{2}{1} \times 0\right) = 192.$$

Summing the calculation for all six inner sums and dividing by $n = 24$ gives our final necklace count: 112720.
Notice that we can make the linear count a special case of the necklace count by adding a single ball of a new colour: $n_{t+1} = 1$. This has the effect of placing a 'cut point' in our circle, making it a line. Correspondingly, in the above calculation the gcd is reduced to 1, and the summation reduces to a single multinomial.

Counting necklaces with n balls (beads) and k colours is a well-studied problem which may be solved using the Orbit Counting Lemma or, more sweepingly, by applying Pólya–Redfield enumeration to derive the appropriate multivariate counting polynomial. Hazel Perfect's formula, published in 1956, provides a direct calculation of individual coefficients in this polynomial.

Web link: teaching.csse.uwa.edu.au/units/CITS7209/polya.pdf

Further reading: *Notes on Counting: An Introduction to Enumerative Combinatorics* by Peter J. Cameron, Cambridge University Press, 2017.