THEOREM OF THE DAY

Moreau's Necklace Formula *Take n balls of t different colours, with $n_i \geq 0$ balls of colour i, $i = 1, \ldots, t$. Then the number of distinct arrangements of the balls, in a line or in a circle, is given, respectively, by:*

Linear: $\displaystyle\binom{n}{n_1, \ldots, n_t} = \frac{n!}{n_1! \cdots n_t!}$; **Circular:** $\displaystyle\frac{1}{n}\sum_{d\,|\,D}\binom{n/d}{n_1/d, \ldots, n_t/d}\varphi(d)$, *where* $D = \gcd(n_1, \ldots n_t)$ *and φ is the Euler totient function (see below left).*

The Euler totient function

For a positive integer n, the Euler totient function, denoted $\varphi(n)$, is defined to be the number of positive integers not exceeding n which are coprime to n. If the distinct primes dividing n are $p_1, p_2, \ldots p_m$ (we may write $p_i\,|\,n, i = 1, \ldots, m$), then the value of $\varphi(n)$ may be calculated explicitly as

$$\varphi(n) = n\left(\frac{p_1 - 1}{p_1}\right)\left(\frac{p_2 - 1}{p_2}\right)\cdots\left(\frac{p_m - 1}{p_m}\right).$$

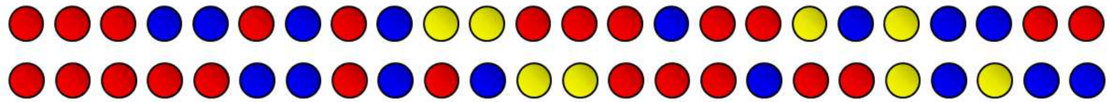
For example,

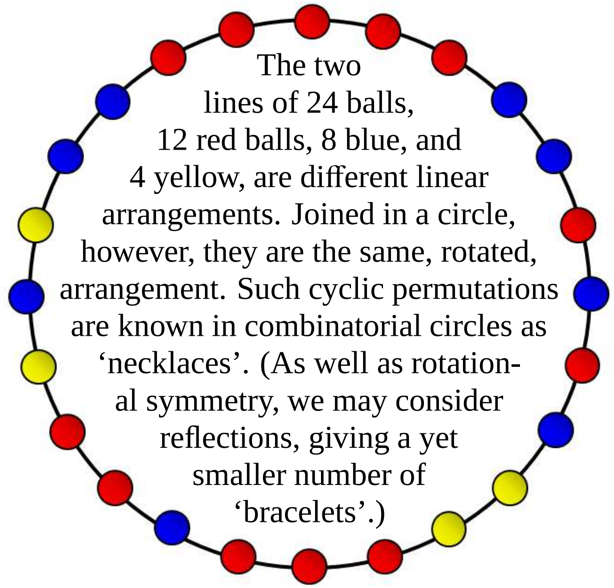
$$\varphi(18) = \varphi(2 \times 3^2) = 18 \times \frac{1}{2} \times \frac{2}{3} = 6.$$

The first few values are tabulated below:

n	1	2	3	4	5	6	7	8	9	10	11
$\varphi(n)$	1	1	2	2	4	2	6	4	6	4	10

the (local) maximum values occurring at the primes.





The two lines of 24 balls, 12 red balls, 8 blue, and 4 yellow, are different linear arrangements. Joined in a circle, however, they are the same, rotated, arrangement. Such cyclic permutations are known in combinatorial circles as 'necklaces'. (As well as rotational symmetry, we may consider reflections, giving a yet smaller number of 'bracelets'.)

In our example there are $n = 24$ balls with $n_1 = 12, n_2 = 8$ and $n_3 = 4$. The multinomial coefficient $\binom{24}{12,8,4}$ gives the number of linear arrangements as a little over 1.3×10^9.

For the necklace (circular) count our sum is over the divisors $\{1, 2, 4\}$ of $D = \gcd(12, 8, 4) = 4$:

$$\frac{1}{24}\left\{\binom{24}{12, 8, 4}\varphi(1) + \binom{12}{6, 4, 2}\varphi(2) + \binom{6}{3, 2, 1}\varphi(4)\right\},$$

giving roughly 5.6×10^7. The first term in the sum accounts for almost all these necklaces: to a first approximation we are removing circular symmetries just by dividing the linear count by the number of balls. Conversely, notice that we can make the linear count a special case of the necklace count by adding a single ball of a new colour: $n_{t+1} = 1$, in any of the 24 possible positions. This has the effect of placing a 'cut point' in our circle, making it a line. Correspondingly, in the above calculation the gcd is reduced to 1, and the summation reduces to a single multinomial.

The entries in the n-th row of Pascal's triangle, beginning with the binomial coefficient $\binom{n}{0}$, sum to 2^n; generalising, the sum of all multinomial coefficients dividing n into t parts is t^n. So if we sum our necklace formula over all possible choices of t colours for our n balls, including cases where some of the n_i are zero, we will get the number of n-ball necklaces having t or fewer colours: $(1/n)\sum_{d|n}\varphi(d)t^{n/d}$. For $n = 4$ and $t = 3$, this evaluates to 24, which you can list, with a little effort! (Click icon, top right, for answer.)

Counting necklaces with n balls (beads) and k colours is a well-studied problem which may be solved using the Orbit Counting Lemma or, more sweepingly, by applying Pólya–Redfield enumeration to derive the appropriate multivariate counting polynomial. Charles Moreau's formula, published in 1872, provides a direct calculation of individual coefficients in this polynomial.

Web link: mathlesstraveled.com/2017/12/12/

Further reading: *Notes on Counting: An Introduction to Enumerative Combinatorics* by Peter J. Cameron, Cambridge University Press, 2017.