Threading Beads – Addendum

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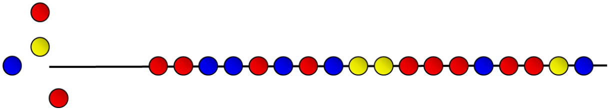
theoremoftheday.org

11 June, 2021

Threading beads

We have n beads of m different colours. In how many ways may we thread them on to a string?



Suppose the distribution of colours is not specified. Then

$$N(n, m) = m \times m \times \ldots \times m = m^n.$$

Threading beads, colour distribution specified

Suppose there are n_i beads of colour i, $n_i \geq 0$, $n_1 + \ldots + n_m = n$.

The number of permutations of n objects of which n_i belong to colour class i is given by the corresponding monomial coefficient:

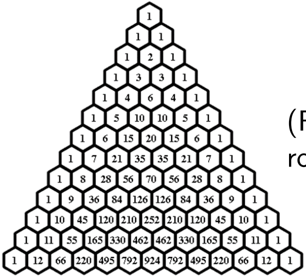
$$N(n; n_1, \ldots, n_m) = \frac{n!}{n_1! \cdots n_m!} = \binom{n}{n_1, \ldots, n_m}.$$

The Multinomial Theorem

$$(x_1 + \ldots + x_m)^n = \sum_{n_1+\ldots+n_m=n} \binom{n}{n_1, \ldots, n_m} x_1^{n_1} x_2^{n_2} \cdots x_m^{n_m}.$$

Putting $x_1 = \ldots = x_m = 1$, we get

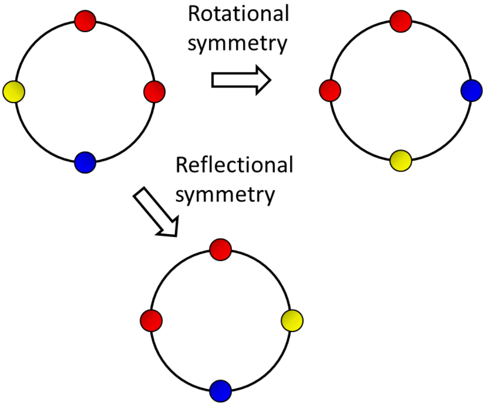
$$m^n = \sum_{n_1+\ldots+n_m=n} \binom{n}{n_1, \ldots, n_m} = N(n, m).$$



(For $m = 2$ this says that the n-th row of Pascal's triangle sums to 2^n.)

Circular bead configurations

Suppose we form our string of beads into a circle and want our count to ignore symmetrical configurations:

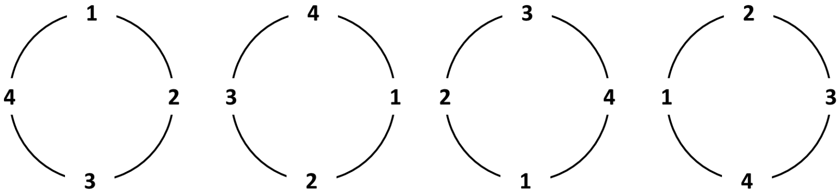


Necklaces

Circular permutations of a string of n coloured beads, counted up to rotational symmetry, are commonly called **necklaces**.

The symmetries are the cyclic group C_n of order n.
E.g.

$$C_4 = \{1,\ (1,2,3,4),\ (1,3)(2,4),\ (1,4,3,2)\}.$$



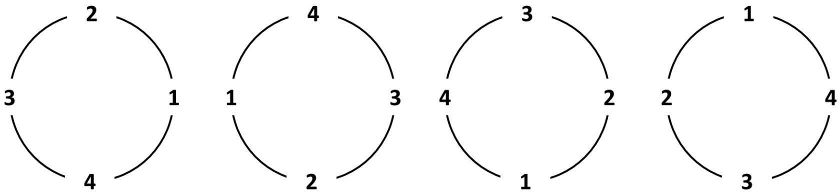
Write $C(n,m)$ for the number of necklaces of n beads using zero or more of each of m possible bead colours.

Bracelets

Circular permutations of a string of n coloured beads, counted up to rotational and reflectional symmetry, are commonly called **bracelets**.

The symmetries are the dihedral group D_{2n} of order $2n$.
E.g.

$$D_8 = C_4 \cup \{(1,2)(3,4),\ (1,4)(2,3),\ (1,3),\ (2,4)\}.$$



A 1956 article

Write $p = p_0 d_0 = p_1 d_1 = p_2 d_2 = \ldots = p_m d_m$, $(p_m = p)$, and similarly etc.

We shall call an arrangement (linear or circular) d_i-symmetrical if it consists of a chain of exactly d_i identical arrangements of $p_i + q_i + \ldots$ objects. (*Example.* *aabaab* is a 2-symmetrical arrangement of the four letters *a* and the two letters *b*.) Clearly every arrangement is d_i-symmetrical for some $i = 0, 1, \ldots, m$.

Denote by K_i the number of linear d_i-symmetrical arrangements; it is easily seen that

$$K_i = \frac{(n/d_i)!}{p_i!\, q_i! \ldots} - \sum_{j=0}^{i-1} S_{ij} K_j,$$

$i = 0, 1, \ldots, m$, where $S_{ij} = 1$ or 0 according as d_i does or does not divide d_j. These equations may be written in the form

$$\begin{aligned}
&K_0 &&= l_0 \\
&S_{10}K_0 + K_1 &&= l_1 \\
&S_{20}K_0 + S_{21}K_1 + K_2 &&= l_2 \\
&\cdots\cdots\cdots\cdots\cdots\cdots && \\
&S_{m0}K_0 + S_{m1}K_1 + S_{m2}K_2 + \ldots + K_m &&= l_m,
\end{aligned}$$

with $l_i = \left(\frac{n}{d_i}\right)! \Big/ p_i!\, q_i! \ldots$; and their solution is

$$K_i = \begin{vmatrix} 1 & 0 & \ldots l_0 & \ldots & 0 \\ S_{10} & 1 & \ldots l_1 & \ldots & 0 \\ S_{20} S_{21} & & \ldots l_2 & \ldots & 0 \\ \cdots & & \cdots & & \\ S_{m0} S_{m1} & & \ldots \underset{(i+1)}{l_m} & \ldots & 1 \end{vmatrix}, \quad \ldots\ldots\ldots (1)$$

$i = 0, 1, \ldots, m$.

Now a d_i-symmetrical circular arrangement gives rise to exactly $\left(\frac{n}{d_i}\right)$ different d_i-symmetrical linear arrangements. Therefore the total number of circular arrangements of the n objects is equal to

$$\sum_{i=0}^{m} \left(\frac{d_i}{n}\right) K_i,$$

where the K_i are given by (1). This is the desired result.

HAZEL PERFECT.

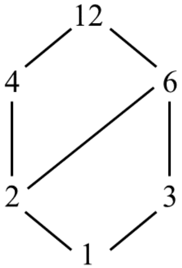
Counts $C(n; n_1, \ldots, n_m)$ necklaces with $n_i \geq 0$ beads of colour i, $n = n_1 + \ldots + n_m$.

Hazel Perfect, "Concerning arrangements in a circle", *The Mathematical Gazette*, Vol. 40, No. 331, 1956, pp. 45-46.

Linear algebra

Perfect's paper solved the counting problem in terms of a set of simultaneous linear equations. Let $M = \gcd(n_1, \ldots n_m)$. Then there are M equations and their matrix is the adjacency matrix of the partially ordered set (poset) of divisors of M.

	1	2	3	4	6	12
1	1	1	1	1	1	1
2		1	0	1	1	1
3			1	0	1	1
4				1	0	1
6					1	1
12						1



The Möbius function for posets

Now the inverse of the adjacency matrix of a poset is the Möbius function

$$\left(\begin{array}{r|rrrrrr} & 1 & 2 & 3 & 4 & 6 & 12 \\ \hline 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & & 1 & 0 & 1 & 1 & 1 \\ 3 & & & 1 & 0 & 1 & 1 \\ 4 & & & & 1 & 0 & 1 \\ 6 & & & & & 1 & 1 \\ 12 & & & & & & 1 \end{array}\right)^{-1} = \begin{array}{r|rrrrrr} & 1 & 2 & 3 & 4 & 6 & 12 \\ \hline 1 & 1 & -1 & -1 & 0 & 1 & 0 \\ 2 & & 1 & 0 & -1 & -1 & 1 \\ 3 & & & 1 & 0 & -1 & 0 \\ 4 & & & & 1 & 0 & -1 \\ 6 & & & & & 1 & -1 \\ 12 & & & & & & 1 \end{array}$$

August Möbius (1790–1868)

For a positive integer n with P distinct prime factors,

$$\mu(n) = \begin{cases} 0 & \text{if any prime factor is squared in } n \\ (-1)^P & \text{otherwise} \end{cases}$$

n	1	2	3	4	5	6	7	8	9	10	11	12
$\mu(n)$	1	−1	−1	0	−1	1	−1	0	0	1	−1	0

A cornerstone of analytic number theory thanks to properties such as

$$\sum_{d \mid n} \mu(d) = 0, \text{ for } n > 1.$$

A digression

M500 Problem 206.6:

Evaluate $\displaystyle\sum_{p\ \text{prime}} \frac{1}{p^2}$ *to any reasonable number of decimal places.*

Solution (ADF):

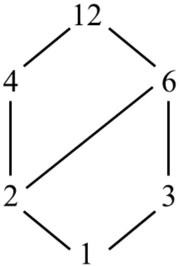
$$\sum_{p\ \text{prime}} \sum_{r=1}^{\infty} \frac{1}{rp^2} \sum_{d\,|\,r} \mu(d) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \log \zeta(2n),$$

$$\text{So:} \quad \sum_{p\ \text{prime}} \frac{1}{p^2} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \log \zeta(2n).$$

Calculating the Möbius function $\mu(x, y)$ for posets

$$\mu(x, y) = \sum_{c} (-1)^{l(c)},$$

where the sum is over all chains c ascending from x to y and $l(c)$ is the number of edges in the chain.



	1	2	3	4	6	12
1	1	-1	-1	0	1	0
2		1	0	-1	-1	1
3			1	0	-1	0
4				1	0	-1
6					1	-1
12						1

E.g., $\mu(3, 12) = 0$ because the two-edge chain $3 - 6 - 12$ contributes $(-1)^2$ to the calculation and the one-edge chain $3 - 12$, not included in the diagram, contributes $(-1)^1$.

Back to Hazel Perfect

We have $n = n_1 + \ldots + n_m$ beads. Suppose $\gcd(n_1, \ldots, n_m) = M$, and suppose the divisors of M are $d_1, \ldots, d_k$.

Perfect derived:

$$C(n; n_1, \ldots, n_m) = \left(\frac{d_1}{n}, \ldots \frac{d_k}{n}\right) \times S^{-1} \times \begin{pmatrix} \binom{n/d_1}{n_1/d_1, \ldots, n_m/d_1} \\ \vdots \\ \binom{n/d_k}{n_1/d_k, \ldots, n_m/d_k} \end{pmatrix},$$

where S is the adjacency matrix of the poset of divisors of M.

Now a d_i-symmetrical circular arrangement gives rise to exactly $\left(\frac{n}{d_i}\right)$ different d_i-symmetrical linear arrangements. Therefore the total number of circular arrangements of the n objects is equal to

$$\sum_{i=0}^{m} \left(\frac{d_i}{n}\right) K_i,$$

where the K_i are given by (1). This is the desired result.

Hazel Perfect plus Möbius

Since S^{-1} is the Möbius function, Perfect's summation can be rewritten as:

$$C(n; n_1, \ldots, n_m) = \frac{1}{n} \sum_{i=1}^{k} \sum_{j=i}^{k} d_i \binom{n/d_j}{n_1/d_j \ldots, n_m/d_j} \mu(d_i, d_j).$$

An example

$n = 8, n_1 = 4, n_2 = 2, n_3 = 2.$

$$M = 2, d_1 = 1, d_2 = 2, \mu = \begin{array}{c|cc} & 1 & 2 \\ \hline 1 & 1 & -1 \\ 2 & & 1 \end{array}$$

$$\begin{aligned} C(8; 4, 2, 2) &= \frac{1}{8} \sum_{i=1}^{2} \sum_{j=i}^{2} d_i \binom{8/d_j}{4/d_j, 2/d_j, 2/d_j} \mu(d_i, d_j) \\ &= \frac{1}{8} \left[\left\{ 1 \times \binom{8}{4, 2, 2} \mu(1, 1) + 1 \times \binom{4}{2, 1, 1} \mu(1, 2) \right\} \right. \\ &\quad \left. + 2 \binom{4}{2, 1, 1} \mu(2, 2) \right] \\ &= \frac{1}{8} [\{1 \times 420 \times 1 + 1 \times 12 \times -1\} + 2 \times 12 \times 1] = 54. \end{aligned}$$

Addendum No. 1

Moreau's Necklace Formula (1872):

$$C(n, m) = \sum_{d \mid m} \varphi(d) n^{m/d}.$$

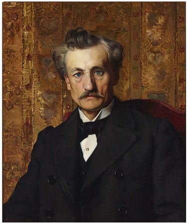
Le colonel Charles Paul Narcisse Moreau (1837–1916). Famous in chess circles for having the worst-ever result in an international tournament, having lost all 26 games at Monte Carol in 1903.

(But he was 65 at the time and the competition involved several of chess history's finest, including Marshall, Pillsbury and Tarrasch.)

Addendum No. 1 continued

Moreau's Necklace Formula:

$$C(n,m) = \sum_{d \mid m} \varphi(d) n^{m/d}.$$



Rediscovered in 1892 by Édouard Jablonski (1848–1923)...

(the portrait is by Michel Richard-Putz).

... and publicised by Percy MacMahon in 1892.

Moreau's Necklace Formula

$$C(n, m) = \sum_{d \mid m} \varphi(d) n^{m/d}.$$

The Euler totient function

For a positive integer n, the Euler totient function, denoted $\varphi(n)$, is defined to be the number of positive integers not exceeding n which are coprime to n. We can calculate φ using the following recursive definition, due to Gauss:

$$\begin{aligned} \varphi(1) &= 1, \\ \varphi(y) &= y - \sum_{x \mid y, x \neq y} \varphi(x), \ y > 1, \end{aligned} \tag{1}$$

The first few values are tabulated below:

n	1	2	3	4	5	6	7	8	9	10	11
$\varphi(n)$	1	1	2	2	4	2	6	4	6	4	10

the (local) maximum values occurring at the primes.

Addendum No. 2

$$\text{Rewrite } C(n; n_1, \ldots, n_m) = \frac{1}{n} \sum_{i=1}^{k} \sum_{j=i}^{k} d_i \binom{n/d_j}{n_1/d_j \ldots, n_m/d_j} \mu(d_i, d_j)$$

$$\text{as } C(n; n_1, \ldots, n_m) = \frac{1}{n} \sum_{t \mid M} \binom{n/t}{n_1/t \ldots, n_m/t} \sum_{s \mid t} s\mu(s, t).$$

e.g. $n = 8, n_1 = 4, n_2 = 2, n_3 = 2.$

$M = 2, d_1 = 1, d_2 = 2, \mu =$

	1	2
1	1	-1
2		1

$$C(8; 4, 2, 2) = \frac{1}{8} \left[\binom{8}{4, 2, 2} \times 1 \times \mu(1, 1) + \binom{4}{2, 1, 1} \{1 \times \mu(1, 2) + 2 \times \mu(2, 2)\} \right]$$

Möbius Inversion

In a finite poset

$$\begin{aligned} f(t) &= \sum_{s \mid t} g(s)\mu(s,t), \text{ for all } t \\ \text{if and only if } g(t) &= \sum_{s \mid t} f(s), \text{ for all } t. \end{aligned}$$

Define $f(t) = \sum_{s \mid t} s\,\mu(s,t)$, for all $t \leq M$. Then $t = \sum_{s \mid t} f(s)$, or $f(t) = t - \sum_{s \mid t,\, s \neq t} f(s)$.

But by Gauss's formula, this is just the definition of φ. So we can rewrite our necklace calculation as a single sum:

$$C(n; n_1, \ldots, n_m) = \frac{1}{n} \sum_{t \mid M} \binom{n/t}{n_1/t, \ldots, n_m/t} \varphi(t).$$

Back in 1872...

$$C(n; n_1, \ldots, n_m) = \frac{1}{n} \sum_{t \mid M} \binom{n/t}{n_1/t, \ldots, n_m/t} \varphi(t).$$

is more or less what le colonel Moreau wrote down in 1872. To derive his necklace formula

$$C(n, m) = \sum_{d \mid m} \varphi(d) n^{m/d}.$$

we must sum over all distributions of colours:

$$C(n, m) = \frac{1}{n} \sum_{n_1 + \ldots + n_m = n} \left(\sum_{t \mid \gcd(n_1, \ldots, n_m)} \binom{n/t}{n_1/t, \ldots, n_m/t} \varphi(t) \right),$$

and use the fact that

$$\sum_{n_1 + \ldots + n_m = n} \binom{n}{n_1, \ldots, n_m} = m^n.$$

Pólya–Redfield Enumeration

The presence of φ in Moreau's formula is reflected in its role in the cycle index of the cyclic group.

Theorem (Redfield, Pólya): The cycle index of the cyclic group C_n is given by

$$Z(C_n) = \frac{1}{n}\sum_{k\,|\,n} \varphi(k) s_k^{n/k},$$

where the s_i are invariants.

To enumerate 3-coloured necklaces, for example, we make a formal substitution $s_k = r^k + b^k + y^k$, where r, b and y are again invariants. For a 4-bead necklace we get:

$$b^4 + b^3 r + b^3 y + 2\,b^2 r^2 + 3\,b^2 ry + 2\,b^2 y^2 + br^3 + 3\,br^2 y$$
$$+3\,bry^2 + by^3 + r^4 + r^3 y + 2\,r^2 y^2 + ry^3 + y^4.$$

The term $3\,b^2 ry$, for example, means there are three necklaces, up to rotational symmetry, having two blue beads, one red bead and one yellow bead.