The Necklace Counting Formula: Addendum

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In M500 Issue 285 I wrote down a formula for the number of necklaces with b_i beads of colour i, $i = 1, \ldots, t$. The formula was given in terms of the partially ordered set (poset) of divisors $d_1, \ldots, d_m$ of the greatest common divisor of the b_i, and of the Möbius function $\mu(x, y)$ for this poset:

$$N(b_1, b_2, \ldots, b_t) = \frac{1}{n} \sum_{i=1}^{m} \sum_{j=i}^{m} d_i \binom{n/d_j}{b_1/d_j, \ldots, b_t/d_j} \mu(d_i, d_j), \tag{1}$$

where $n = b_1 + \ldots + b_t$. The formula works if some of the b_i are zero, using the fact that $\gcd(x, 0) = x$. The multinomial coefficient $\binom{x}{y_1, \ldots, y_t}$ is evaluated as $x!/(y_1! \cdots y_t!)$ and counts the number of non-circular permutations of x objects of which y_1 are colour 1, y_2 are colour 2, etc. The usual binomial coefficient $\binom{x}{y}$ is a short way of writing $\binom{x}{y, x-y}$. Recall that if we sum these binomial coefficients over all choices of y the result is 2^x and this generalises to monomial coefficients:

$$\sum_{y_1+\ldots+y_t=x} \binom{x}{y_1, \ldots, y_t} = t^x. \tag{2}$$

By 'necklace' we mean a circular permutation of objects belonging to distinguished classes (colours) taking into consideration rotational symmetry. Formula (1) was derived from a 1956 *Mathematical Gazette* paper in which the number of such permutations was found as the solution to a set of simultaneous equations. I found the double sum above to be preferable to saying "the solution is now found by inverting the equation matrix". However, I didn't go far enough: a single summation was just around the corner!

I also suggested that it was textbook stuff to specify a formula for the number of permutations with the number of colours, t, specified but with no restriction on their distribution. This is true, the formula is the following

$$N(n, t) = \frac{1}{n} \sum_{d \le n} \varphi(d) t^{n/d}, \tag{3}$$

using poset notation $d \le n$ to mean d divides into n. The function φ is Euler's totient function whose value at a positive integer x is the number of positive integers less than x and coprime to x. For example,

$$\begin{aligned} N(6, 3) &= \frac{1}{6}\left(\varphi(1) \times 3^6 + \varphi(2) \times 3^3 + \varphi(3) \times 3^2 + \varphi(6) \times 3^1\right) \\ &= \frac{1}{6}(1 \times 729 + 1 \times 27 + 2 \times 9 + 2 \times 3) = 130. \end{aligned}$$

There are many ways to determine the values of $\varphi(x)$. We are going to find convenient the following recursive definition, due to Gauss:

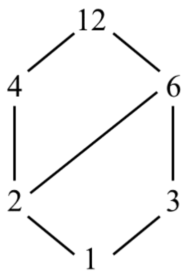
$$\begin{aligned} \varphi(1) &= 1, \\ \varphi(y) &= y - \sum_{x<y} \varphi(x), \; y > 1, \end{aligned} \tag{4}$$

where we are still using poset notation and '<' means 'strictly divides'.

Formula (3) is sometimes misattributed to Captain Percy Alexander MacMahon who indeed wrote about it in 1892 but acknowledged its prior discovery by another soldier, Monsieur le Colonel Charles

Paul Narcisse Moreau. Moreau had solved our counting problem in 1872 and was, as far as I know, the first to do so. Formula (3) is a special case of his solution and I would now like to explain how we get to it from formula (1).

Let me recall what, in my original contribution, I said about the Möbius function. We are concerned with the version for the poset of divisors, as illustrated below left. Somewhat informally the value of the Möbius function $\mu(x,y)$ for two elements x and y of this poset is $\sum(-1)^{l(c)}$ where the sum is over all upward 'chains' c from x to y and $l(c)$ is the number of edges in the chain. The diagram of the poset only shows 'immediate' division but our summation must also include all implied edges, such as the edge from 3 to 12.



	1	2	3	4	6	12
1	1	−1	−1	0	1	0
2		1	0	−1	−1	1
3			1	0	−1	0
4				1	0	−1
6					1	−1
12						1

The values of $\mu(x,y)$ for the poset on the left. For example, $\mu(3,12) = 0$ because there are two chains from 3 to 12: the two-edge chain $3 - 6 - 12$ which contributes $(-1)^2$ to the calculation and the one-edge chain $3 - 12$, not included in the diagram, which contributes $(-1)^1$.

I will rewrite formula (1) in a more poset-friendly form:

$$N(b_1,b_2,\ldots,b_t) = \frac{1}{n}\sum_{e\leq M}\left\{\binom{n/e}{n_1/e,\ldots,n_t/e}\sum_{d\leq e} d\,\mu(d,e)\right\}, \tag{5}$$

where $M = \gcd(b_1,\ldots,b_t)$ and I no longer need to list the divisors of M explicitly because the poset notation $e \leq M$ takes care of that. The order of summation has changed from (1) but that is just a matter of counting by column instead of by row. The important thing is to isolate the sum $\sum_{d\leq e} d\,\mu(d,e)$, because such a sum is amenable to Möbius inversion, a contribution to number theory by August Ferdinand Möbius in the 1830s, transferred to posets by group theorists in the 1930s as follows:

$$g(y) = \sum_{x\leq y} f(x),\ \text{for all } y, \text{ if and only if }\ f(y) = \sum_{x\leq y} g(x)\mu(x,y),\ \text{for all } y.$$

Define $f(e) = \sum_{d\leq e} d\,\mu(d,e)$, for all $e \leq M$. Then $e = \sum_{d\leq e} f(d)$, or $f(e) = e - \sum_{d<e} f(e)$. And now from Gauss's formula (4):

$$N(b_1,b_2,\ldots,b_t) = \frac{1}{n}\sum_{e\leq M}\binom{n/e}{n_1/e,\ldots,n_t/e}\varphi(e). \tag{6}$$

This is what Colonel Moreau wrote down almost 150 years ago, and I should have written down, two and half years ago, for M500. Suppose we now sum over all possible distributions of the n beads amoung the t colours. We get

$$N(n,t) = \frac{1}{n}\sum_{b_1+\ldots+b_t=n}\left(\sum_{e\leq\gcd(b_1,\ldots,b_m)}\binom{n/e}{b_1/e,\ldots,b_t/e}\varphi(e)\right).$$

This double sum groups the monomial terms according partitions of n; we will get the same result if we group according to divisors of n:

$$N(n,t) = \frac{1}{n}\sum_{e\leq n}\left(\sum_{(b'_1+\ldots+b'_t)e=n}\binom{n/e}{b'_1,\ldots,b'_t}\varphi(e)\right) = \frac{1}{n}\sum_{e\leq n}\left(\varphi(e)\sum_{b'_1+\ldots+b'_t=n/e}\binom{n/e}{b'_1,\ldots,b'_t}\right),$$

and we recover Moreau's formula (3) by applying identity (2).