Binomial coefficients

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1										
1	1									
1	2	1								
1	3	3	1							
1	4	6	4	1						
1	5	10	10	5	1					
1	6	15	20	15	6	1				
1	7	21	35	35	21	7	1			
1	8	28	56	70	56	28	8	1		
1	9	36	84	126	126	84	36	9	1	
1	10	45	120	210	252	210	120	45	10	1

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Binomial coefficients $\binom{n}{m}$

Defined by $\binom{n}{m}$ = number of ways of choosing m objects from n. Same as number of m−subsets of an n-set.

Also: defined for $1 \leq m < n$, by $\binom{n}{m} = \binom{n-1}{m} + \binom{n-1}{m-1}$, and by $\binom{n}{n} = \binom{n}{0} = 1$.

Also: defined by $\binom{n}{m} = \frac{n!}{m!(n-m)!} = \frac{n\times(n-1)\times\cdots\times(n-m+1)}{m\times(m-1)\times\cdots\times 2\times 1}$, which remains defined for any real or even complex n, consistent with the binomial theorem:

$$(x+y)^n = \sum_{k=0}^{\infty} \binom{n}{k} x^{n-k}\, y^k, \ \binom{n}{k} = 0, k > n.$$

1										
1	1									
1	2	1								
1	3	3	1							
1	4	6	4	1						
1	5	10	10	5	1					
1	6	15	20	15	6	1				
1	7	21	35	35	21	7	1			
1	8	28	56	70	56	28	8	1		
1	9	36	84	126	126	84	36	9	1	
1	10	45	120	210	252	210	120	45	10	1

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We may prefer to use the first definition for (combinatorial, more intuitive) proofs of binomial identities.
E.g. sum of row n entries is 2^n.
Easy but unrevealing algebraic proof vs ‘obvious’ combinatorial proof.

The hockey stick identity

	0	1	2	3	4	5	6	7	8	...
0	1									
1	1	1								
2	1	2	1							
3	1	3	3	1						
4	1	4	6	4	1					
5	1	5	10	10	5	1				
6	1	6	15	20	15	6	1			
7	1	7	21	35	35	21	7	1		
8	1	8	28	56	70	56	28	8	1	
⋮										

$$\binom{n+1}{m+1} = \sum_{r=m}^{n} \binom{r}{m}$$

Combinatorial proof:

$$\binom{n+1}{m+1}$$

$$= \binom{n}{m}$$

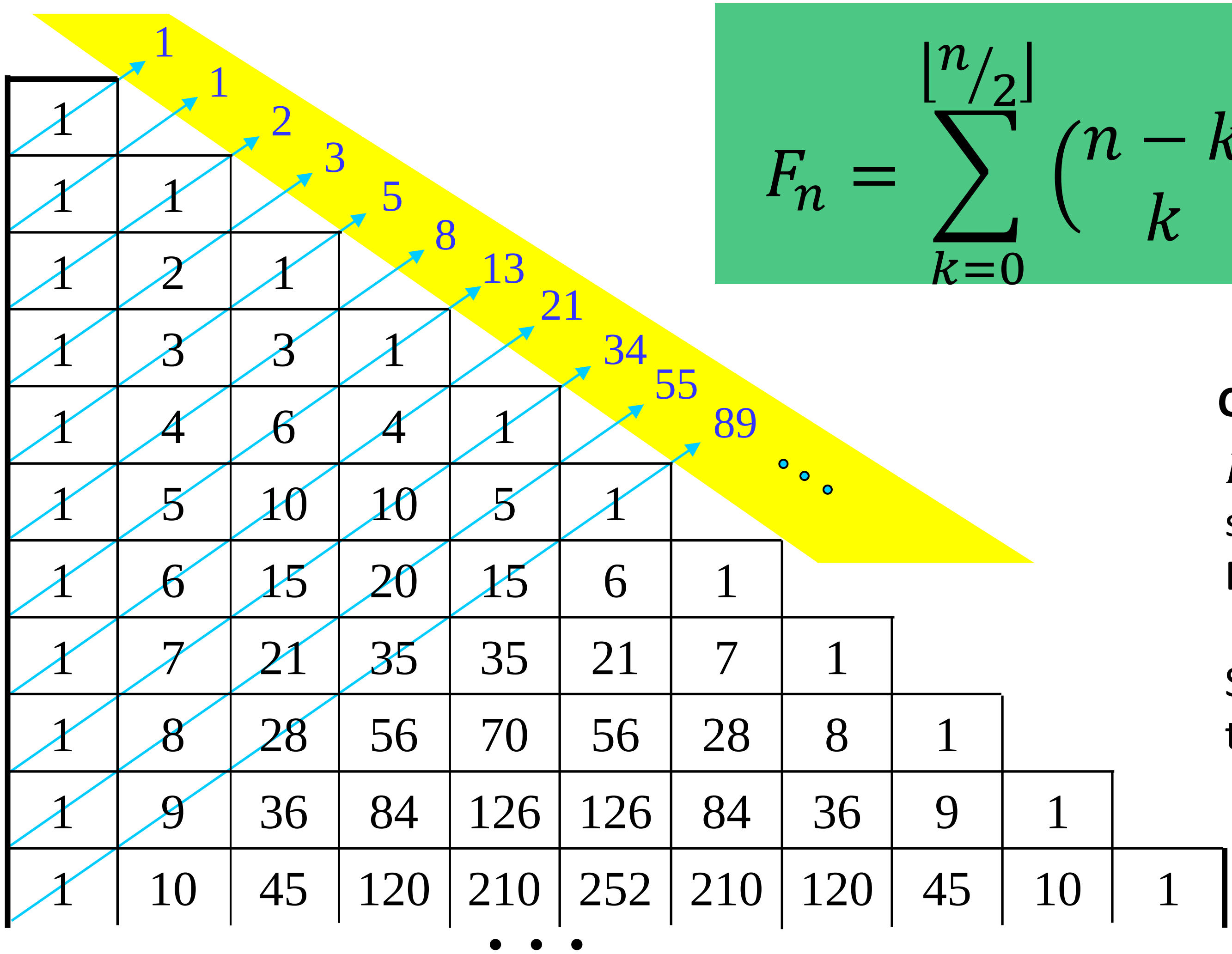
$$+ \binom{n-1}{m}$$

$$+ \binom{n-2}{m}$$

$$+ \binom{n-3}{m} + \ \ldots$$

Fibonacci and Pascal

Sums of ‘diagonals’ are **Fibonacci numbers**



$$F_n = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k}$$

$$F_5 = 8 = \binom{5}{0} + \binom{4}{1} + \binom{3}{2}$$

Combinatorial proof:

F_n is the number of ways to write n as a sum of 1s and 2s.
E.g. $4 = 1\,1\,1\,1\,1,\ \ 1\,1\,2,\ \ 1\,2\,1,\ \ 2\,1\,1,\ \ 2\,2$

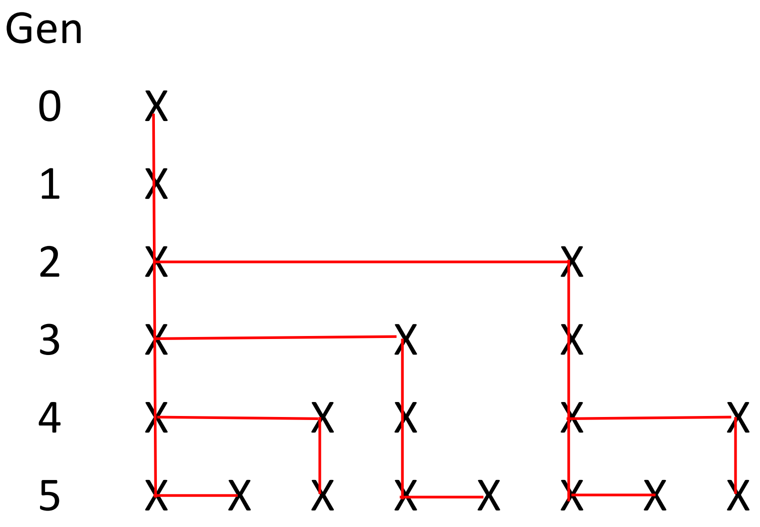
So we must choose up to $\lfloor n/2 \rfloor$ positions for the 2s

Fibonacci and rabbits

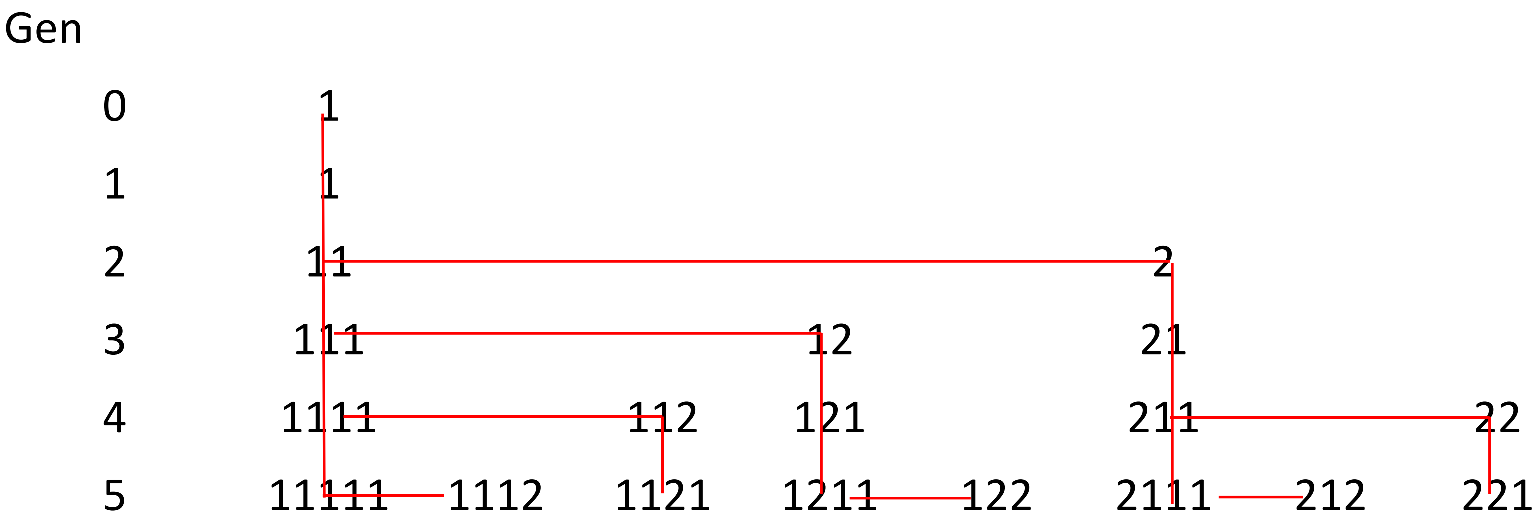
F_n is the number of pairs of breeding rabbits at generation n, motivated by the recurrence $F_n = F_{n-1} + F_{n-2}$.

But why is F_n is the number of ways to write n as a sum of 1s and 2s?

A bijection between generations of pairs of rabbits and 1-2 sequences completes the combinatorial proof of Pascal vs Fibonacci:

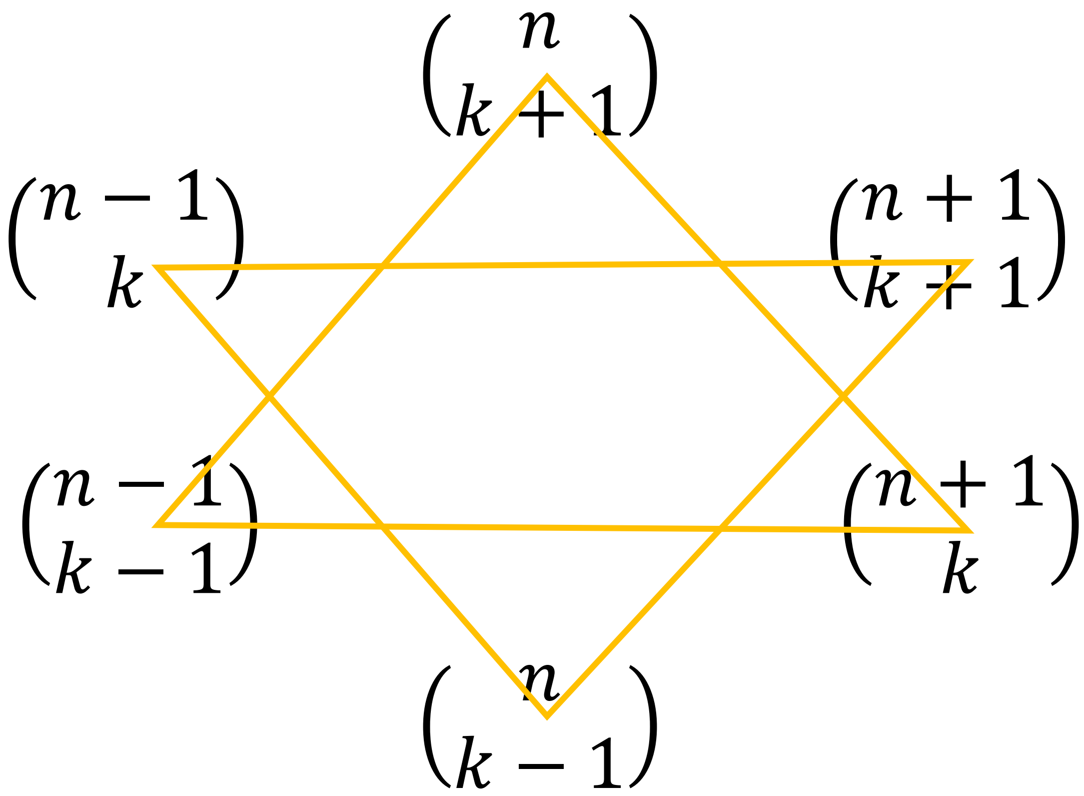


Rabbits



1-2 sequences

The ‘Star of David’ theorem



$$\binom{n-1}{k} \times \binom{n+1}{k+1} \times \binom{n}{k-1}$$
$$= \binom{n}{k+1} \times \binom{n-1}{k-1} \times \binom{n+1}{k}$$

1										
1	1									
1	2	1								
1	3	3	1							
1	4	6	4	1						
1	5	10	10	5	1					
1	6	15	20	15	6	1				
1	7	21	35	35	21	7	1			
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Can we find a combinatorial explanation for equalities of triple products of binomial coefficients?? Note the closely related results that says the triples have equal GCDs, also (I think) lacking a combinatorial proof.

Harlan J. Brothers' formula

A relationship between Pascal and Euler's number e appears to have first been discovered by Harlan J. Brothers in 2012.Let s_n = product of entries in row n of triangle. Then

$$\lim_{n\to\infty} \frac{s_{n-1}s_{n+1}}{s_n^2} = e.$$

1	**1**										
1	1	**1**									
1	2	1	**2**								
1	3	3	1	**9**							
1	4	6	4	1	**96**						
1	5	10	10	5	1	**2500**					
1	6	15	20	15	6	1	**162000**				
1	7	21	35	35	21	7	1	**26471025**			
1	8	28	56	70	56	28	8	1	**11014635520**		
1	9	36	84	126	126	84	36	9	1	**11759522374656**	
1	10	45	120	210	252	210	120	45	10	1	**3240609120000000**

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$$\frac{s_7 \times s_9}{s_8^2} = \frac{26471025 \times 11759522374656}{11014635520^{\ 2}} = \frac{311286610767578342400}{121322195638445670400} \approx 2.5658$$

$$\frac{s_{99} \times s_{101}}{s_{100}^2} \approx 2.7048.$$

Algebraic proof:
a calculation shows that $\frac{s_{n-1}s_{n+1}}{s_n^2} = \left(1+\frac{1}{n}\right)^n$ which in the limit is equal to e.

Combinatorial proof:
???