THEOREM OF THE DAY

Moessner's Magic *For integers $n \geq 1$ and $k \geq 2$, the value of n^k is given by*

$$\sum_{r=1}^{n}\sum_{c=0}^{k-2}((n-r)k+c+1)\,(r-1)^c r^{k-2-c}.$$

Moessner's magic (algorithmic version): the following applies to every integer $k \geq 2$: if you cross out every k-th number from the series of natural numbers and form the sum series from the remaining numbers; then cross out every $(k-1)$-th number from this and form the sum series again, then cross out every $(k-2)$-th number from this and form the sum series again, and continue this process until you finally cross out every second number at the $(k-1)$-th step and then form the sum series, this creates the series of k-th powers $1^k, 2^k, 3^k, 4^k, \ldots$.

E.g. $k = 3$:										
	1	2	~~3~~	4	5	~~6~~	7	8	~~9~~	10
	1	~~3~~		7	~~12~~		19	~~27~~		37
	1			8			27			64

Alfred Moessner presented his method in 1951. For John Conway and Richard Guy it deserved the name 'magic', partly perhaps for its having apparently eluded centuries of mathematicians such as Euler and Jacobi. Although described here as a surprising formula for calculating integer powers it has been more widely treated as a 'sieve' which may be generalised to produce many different arithmetic sequences without explicit use of multiplication.

Web link: thatsmaths.com/2017/09/14/moessners-magical-method

Further reading: The Book of Numbers by John H. Conway and Richard K. Guy, Copernicus, 1996, chapter 3.

The sum series of a sequence with every kth entry deleted may be calculated by multiplying the sequence (as a column vector) by a suitable matrix, as illustrated on the right. The repeated process of deletion and summing in Moessner's algorithm then becomes a product of matrices. Below, this is shown schematically for $k = 5$. The matrix that deletes every kth entry will have a number of rows that is a fraction $(k-1)/k$ of its number of columns. So the number of columns of each matrix must be chosen to match the number of rows of the following matrix.

$$\begin{bmatrix} 1&0&0&0&0&0&0&0&0&0\\ 1&1&0&0&0&0&0&0&0&0\\ 1&1&0&1&0&0&0&0&0&0\\ 1&1&0&1&1&0&0&0&0&0\\ 1&1&0&1&1&0&1&0&0&0\\ 1&1&0&1&1&0&1&1&0&0\\ 1&1&0&1&1&0&1&1&0&1 \end{bmatrix} \times \begin{bmatrix}1\\2\\3\\4\\5\\6\\7\\8\\9\\10\end{bmatrix} = \begin{bmatrix}1\\3\\7\\12\\19\\27\\37\end{bmatrix}$$

$$\underset{\text{delete every 2nd}}{6\times 12} \times \underset{\text{delete every 3rd}}{12\times 18} \times \underset{\text{delete every 4th}}{18\times 24} \times \underset{\text{delete every 5th}}{24\times 30} \times 30\times 1$$

The interest in these matrix products is that they produce what we may call "Moessner matrices" which have a beautifully simple structure. This structure, depicted below for our $k = 5$ schema, easily translates into the double sum of our theorem which, although a bit cumbersome, magically achieves calculation of kth powers using powers no higher than the $(k-2)$-th.

$$\begin{bmatrix}
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
8&4&2&1&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
27&18&12&8&0&8&4&2&1&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
64&48&36&27&0&27&18&12&8&0&8&4&2&1&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
125&100&80&64&0&64&48&36&27&0&27&18&12&8&0&8&4&2&1&0&1&0&0&0&0&0&0&0&0&0\\
216&180&150&125&0&125&100&80&64&0&64&48&36&27&0&27&18&12&8&0&8&4&2&1&0&1&0&0&0&0
\end{bmatrix}$$