Sums of kth powers of integers

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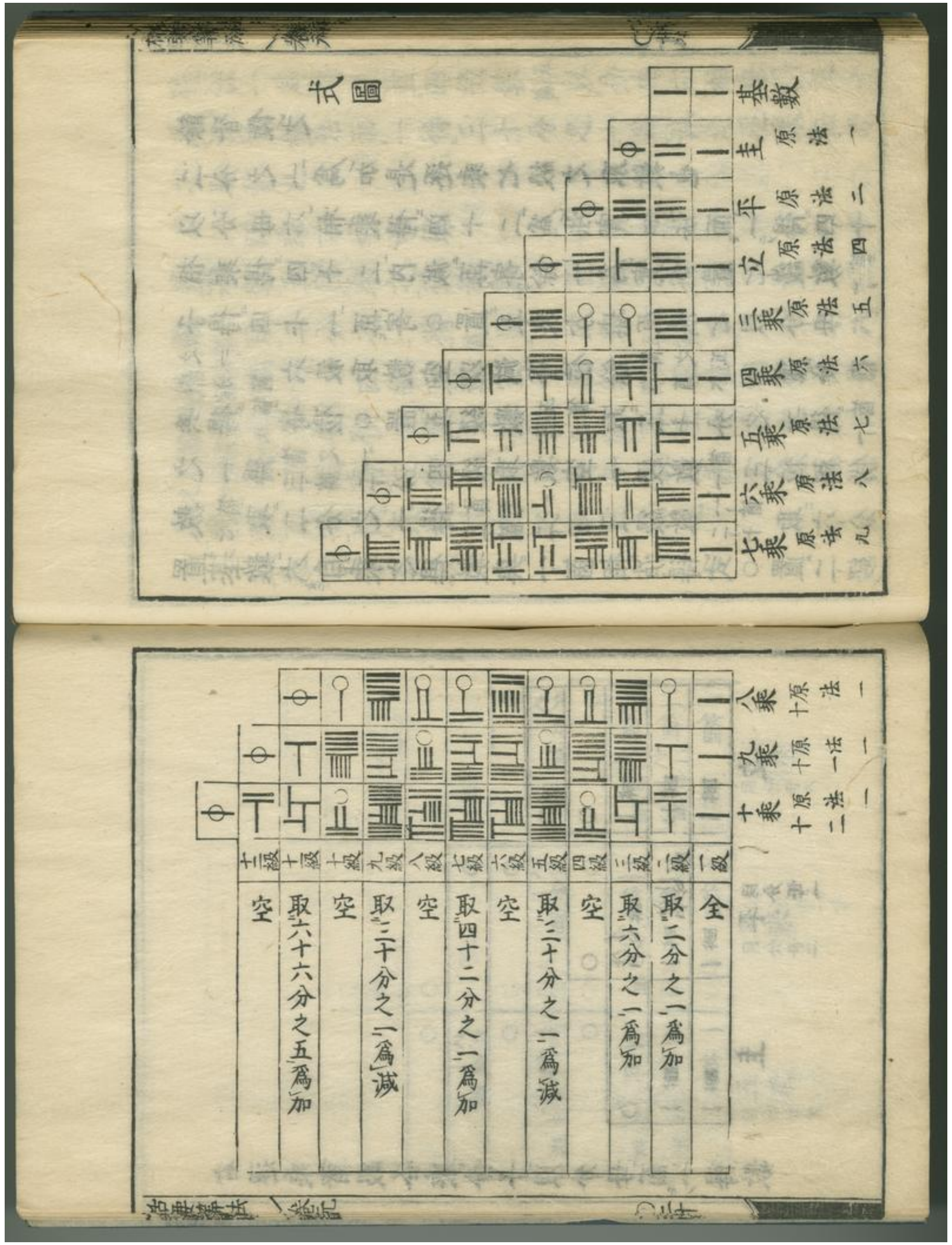
LSBU Maths Study Group
7th August 2025



$$1+2+3+\ldots+n$$

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Maths in a minute: Shake to solve

31 May, 2013

Suppose you have $n+1$ people in a room and each person shakes hands with each other person once. How many handshakes do you get in total? The first person shakes hands with n other people, the second shakes hands with the $n-1$ remaining people, the third shakes hands with $n-2$ remaining people, etc, giving a total of $n+(n-1)+(n-2)+\ldots+2+1$ handshakes.

But we can also look at this in another way: each person shakes hands with n others and there are $n+1$ people, giving $n \times (n+1)$ handshakes. But this counts every handshake twice, so we need to divide by 2, giving a total of

$$\frac{n \times (n+1)}{2}$$

handshakes. Putting these two arguments together, we have just come up with the formula for summing the first n integers and we've proved that it is correct:

$$n+(n-1)+(n-2)+\ldots+2+1=\frac{n \times (n+1)}{2}.$$

Maths can be so easy!

$$1^3 + 2^3 + 3^3 + \ldots + n^3$$

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Sums of Powers of Positive Integers - Aryabhata (b. 476), northern India

Author(s): Janet Beery (University of Redlands)

The northern Indian mathematician and astronomer, Aryabhata, born in 476, wrote one of the earliest known Indian mathematics and astronomy books, the *Aryabhatiya*, in 499 (Katz, p. 212). In Section II, Stanza 22, of the *Aryabhatiya*, he wrote:

> *"The sixth part of the product of three quantities consisting of the number of terms, the number of terms plus one, and twice the number of terms plus one is the sum of the squares. The square of the sum of the (original) series is the sum of the cubes. (Katz, 217)"*

The first sentence gives the formula from pages 1 and 3, above, for the sum of the squares; the second says, in our notation, that

$$(1+2+3+\cdots+n)^2 = 1^3+2^3+3^3+\cdots+n^3.$$

If we replace $1+2+3+\cdots+n$ by $\frac{n(n+1)}{2}$, we obtain

$$1^3+2^3+3^3+\cdots+n^3 = \left(\frac{n(n+1)}{2}\right)^2.$$

Notice a kind of 'quadratic bootstrapping' from power k = 1 to power k = 3

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Linear bootstrapping

Cornell University

We gratefully acknowledge supp
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arXiv > math > arXiv:2203.13870v1

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Mathematics > General Mathematics

[Submitted on 25 Mar 2022]

A simple mnemonic to compute sums of powers

Alessandro Mariani

We give a simple recursive formula to obtain the general sum of the first N natural numbers to the rth power. Our method allows one to obtain the general formula for the $(r+1)$th power once one knows the general formula for the rth power. The method is very simple to remember owing to an analogy with differentiation and integration. Unlike previously known methods, no knowledge of additional specific constants (such as the Bernoulli numbers) is needed. This makes it particularly suitable for applications in cases when one cannot consult external references, for example mathematics competitions.

Subjects: General Mathematics (math.GM)
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$$\sum_{n=1}^{N} n = \frac{1}{2}N(N+1)\ , \qquad (10)$$

which is the correct formula. Now for $r = 2$ we have

$$\frac{d}{dN}\sum_{n=1}^{N} n^2 = 2\sum_{n=1}^{N} n = N^2 + N\ . \qquad (11)$$

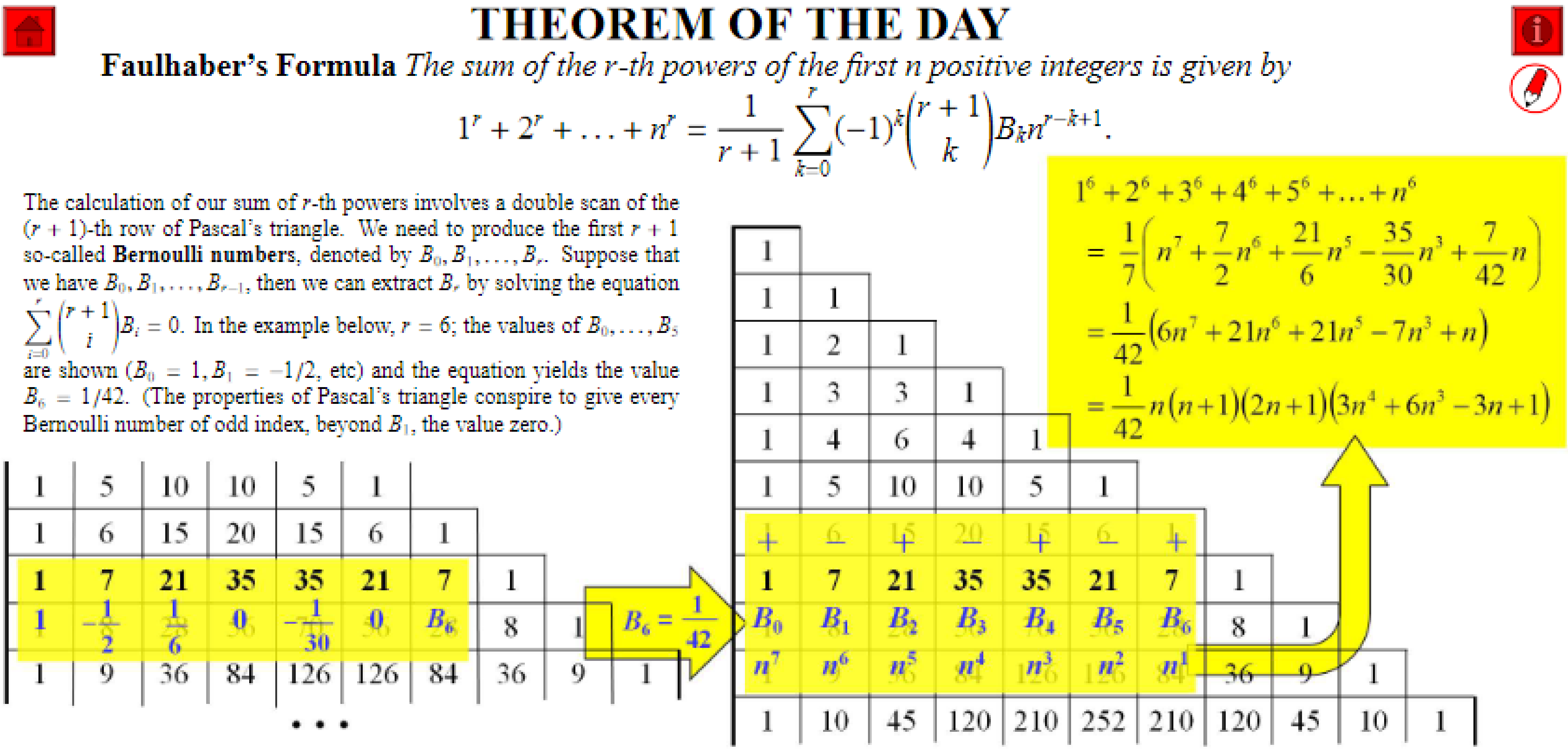
Integrating and adding a linear term

$$\sum_{n=1}^{N} n^2 = \frac{1}{3}N^3 + \frac{1}{2}N^2 + CN\ . \qquad (12)$$

Substituting $N = 1$ we have $1/3 + 1/2 + C = 1$, that is $C = 1/6$, so that

$$\sum_{n=1}^{N} n^2 = \frac{1}{3}N^3 + \frac{1}{2}N^2 + \frac{1}{6}N\ . \qquad (13)$$

Non-bootstrapping? The Bernoulli numbers



(A famous simultaneity: Seki and Bernoulli)

$\int n = \frac{1}{2}nn + \frac{1}{2}n$
$\int nn = \frac{1}{3}n^3 + \frac{1}{2}nn + \frac{1}{6}n$
$\int n^3 = \frac{1}{4}n^4 + \frac{1}{2}n^3 + \frac{1}{4}nn$
$\int n^4 = \frac{1}{5}n^5 + \frac{1}{2}n^4 + \frac{1}{3}n^3 - \frac{1}{30}n$
$\int n^5 = \frac{1}{6}n^6 + \frac{1}{2}n^5 + \frac{5}{12}n^4 - \frac{1}{12}nn$
$\int n^6 = \frac{1}{7}n^7 + \frac{1}{2}n^6 + \frac{1}{2}n^5 - \frac{1}{6}n^3 + \frac{1}{42}n$
$\int n^7 = \frac{1}{8}n^8 + \frac{1}{2}n^7 + \frac{7}{12}n^6 - \frac{7}{24}n^4 + \frac{1}{12}nn$
$\int n^8 = \frac{1}{9}n^9 + \frac{1}{2}n^8 + \frac{2}{3}n^7 - \frac{7}{15}n^5 + \frac{2}{9}n^3 - \frac{1}{30}n$
$\int n^9 = \frac{1}{10}n^{10} + \frac{1}{2}n^9 + \frac{3}{4}n^8 - \frac{7}{10}n^6 + \frac{1}{2}n^4 - \frac{1}{12}nn$
$\int n^{10} = \frac{1}{11}n^{11} + \frac{1}{2}n^{10} + \frac{5}{6}n^9 - 1n^7 + 1n^5 - \frac{1}{2}n^3 + \frac{5}{66}n$

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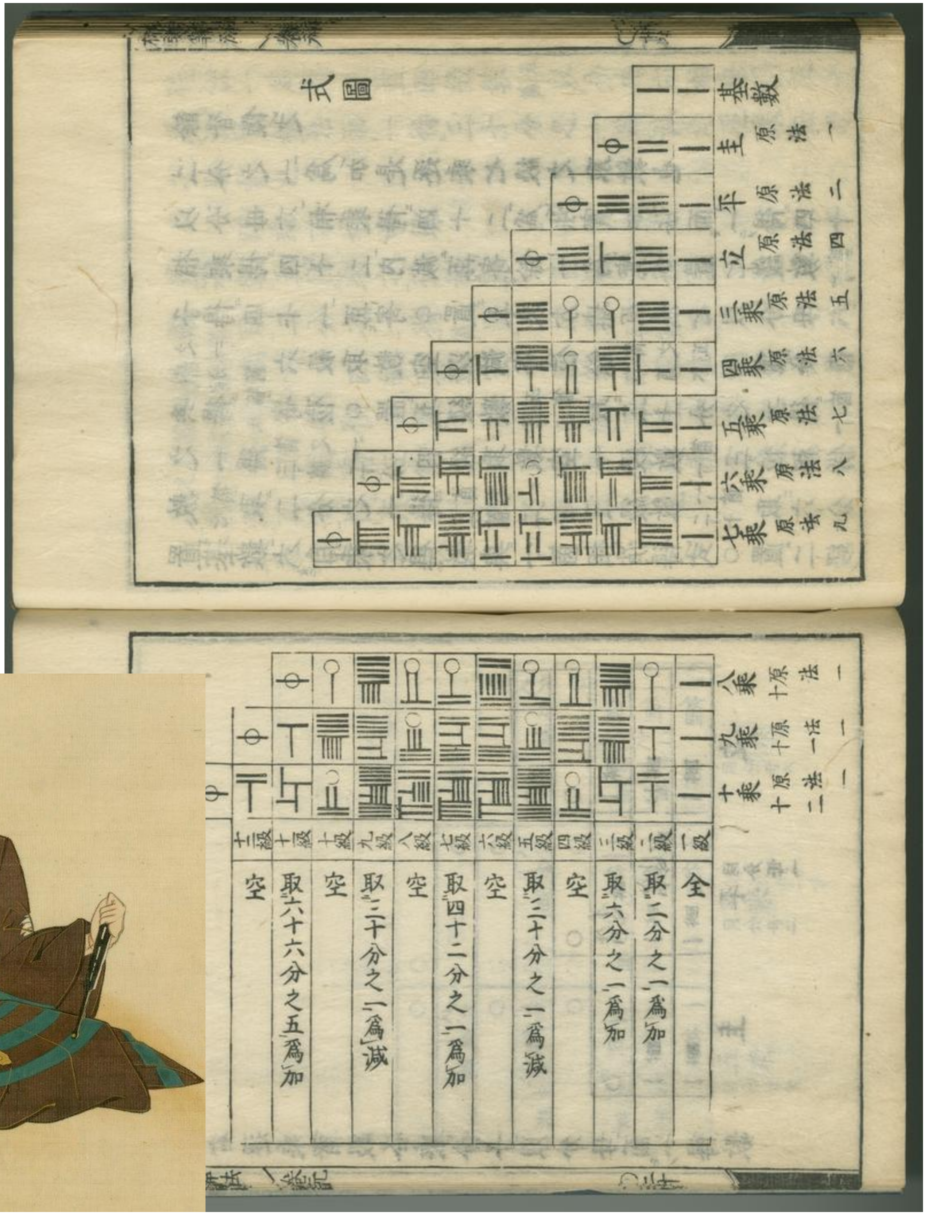
$$\int n^c = \frac{1}{c+1}n^{c+1} + \frac{1}{2}n^c + \frac{c}{2}An^{c-1} + \frac{c \cdot c-1 \cdot c-2}{2 \cdot 3 \cdot 4}Bn^{c-}$$
$$+ \frac{c \cdot c-1 \cdot c-2 \cdot c-3 \cdot c-4}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6}Cn^{c-5}$$
$$+ \frac{c \cdot c-1 \cdot c-2 \cdot c-3 \cdot c-4 \cdot c-5 \cdot c-6}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8}Dn^{c-7} \cdots$$ & ita deinceps,

exponentem potestatis ipsius n continué minuendo binario, quosque perveniatur ad n vel nn. Literae capitales A, B, C, D & c. ordine denotant coëfficientes ultimorum terminorum pro $\int nn$, $\int n^4$, $\int n^6$, $\int n^8$, & c. nempe

$$A = \frac{1}{6}, B = -\frac{1}{30}, C = \frac{1}{42}, D = -\frac{1}{30}.$$



Jakob Bernoulli's *Ars Conjectandi* (1713)

Seki Takakazu's *Katsuyō Sanpō* (1712)

Faulhaber as a linear transform

Construct a lower triangular matrix *Q* from Pascal's triangle.

1										
1	1									
1	2	1								
1	3	3	1							
1	4	6	4	1						
1	5	10	10	5	1					
1	6	15	20	15	6	1				
1	7	21	35	35	21	7	1			
1	8	28	56	70	56	28	8	1		
1	9	36	84	126	126	84	36	9	1	
1	10	45	120	210	252	210	120	45	10	1

• • •

	1	2	3	4	5	6	…
0	0						
1	0	0					
2	0	1	0				
3	0	2	2	0			
4	0	3	5	3	0		
5	0	4	9	9	4	0	
6	0	5	14	19	14	5	0
⋮							

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 & 0 \\ 3 & 5 & 3 & 0 & 0 \\ 4 & 9 & 9 & 4 & 0 \\ 5 & 14 & 19 & 14 & 5 \end{bmatrix}$$

Inverting (infinite) lower triangular matrices

$$X[[n]|[n]]^{-1} = X^{-1}[[n]|[n]]$$
$$[n] = \{1, \dots, n\}$$

$$(a)^{-1} = \left(\frac{1}{a}\right)$$

$$\begin{pmatrix} a & 0 \\ b & c \end{pmatrix}^{-1} = \frac{1}{ac}\begin{pmatrix} c & 0 \\ -b & a \end{pmatrix} = \begin{pmatrix} \frac{1}{a} & 0 \\ -\frac{b}{ac} & \frac{1}{c} \end{pmatrix}$$

$$\begin{pmatrix} a & 0 & 0 \\ b & c & 0 \\ d & e & f \end{pmatrix}^{-1} = \frac{1}{acf}\begin{pmatrix} cf & 0 & 0 \\ -bf & af & 0 \\ be-cd & -ae & ac \end{pmatrix} = \begin{pmatrix} \frac{1}{a} & 0 & 0 \\ -\frac{b}{ac} & \frac{1}{c} & 0 \\ \frac{be-cd}{acf} & -\frac{e}{cf} & \frac{1}{f} \end{pmatrix}$$

Inverting Pascal

$$\begin{bmatrix} 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 3 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 \\ 1 & 3 & 3 & 1 & 0 \\ 1 & 4 & 6 & 4 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ -1 & 3 & -3 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 \\ -1 & 3 & -3 & 1 & 0 \\ 1 & -4 & 6 & -4 & 1 \end{bmatrix}$$

$$P_{ij}{}^{-1} = (-1)^{i+j} P_{ij}$$

Inverting Pascal minus 1

$$[\,1\,],\ \begin{bmatrix}1&0\\2&2\end{bmatrix},\ \begin{bmatrix}1&0&0\\2&2&0\\3&5&3\end{bmatrix},\ \begin{bmatrix}1&0&0&0\\2&2&0&0\\3&5&3&0\\4&9&9&4\end{bmatrix},\ \begin{bmatrix}1&0&0&0&0\\2&2&0&0&0\\3&5&3&0&0\\4&9&9&4&0\\5&14&19&14&5\end{bmatrix}$$

$$[\,1\,],\ \begin{bmatrix}1&0\\-1&\frac{1}{2}\end{bmatrix},\ \begin{bmatrix}1&0&0\\-1&\frac{1}{2}&0\\\frac{2}{3}&-\frac{5}{6}&\frac{1}{3}\end{bmatrix},\ \begin{bmatrix}1&0&0&0\\-1&\frac{1}{2}&0&0\\\frac{2}{3}&-\frac{5}{6}&\frac{1}{3}&0\\-\frac{1}{4}&\frac{3}{4}&-\frac{3}{4}&\frac{1}{4}\end{bmatrix},\ \begin{bmatrix}1&0&0&0&0\\-1&\frac{1}{2}&0&0&0\\\frac{2}{3}&-\frac{5}{6}&\frac{1}{3}&0&0\\-\frac{1}{4}&\frac{3}{4}&-\frac{3}{4}&\frac{1}{4}&0\\-\frac{1}{30}&-\frac{1}{3}&\frac{5}{6}&-\frac{7}{10}&\frac{1}{5}\end{bmatrix}$$

$${Q_{ij}}^{-1} = ?$$

Inverting Pascal minus 1 and summing rows

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{2}{3} & -\frac{5}{6} & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{4} & \frac{3}{4} & -\frac{3}{4} & \frac{1}{4} & 0 & 0 & 0 & 0 \\ -\frac{1}{30} & -\frac{1}{3} & \frac{5}{6} & -\frac{7}{10} & \frac{1}{5} & 0 & 0 & 0 \\ \frac{1}{12} & -\frac{1}{12} & -\frac{5}{12} & \frac{11}{12} & -\frac{2}{3} & \frac{1}{6} & 0 & 0 \\ \frac{1}{42} & \frac{1}{6} & -\frac{1}{6} & -\frac{1}{2} & 1 & -\frac{9}{14} & \frac{1}{7} & 0 \\ -\frac{1}{12} & \frac{1}{12} & \frac{7}{24} & -\frac{7}{24} & -\frac{7}{12} & \frac{13}{12} & -\frac{5}{8} & \frac{1}{8} \end{bmatrix} \quad \begin{matrix} 1 \\ -\frac{1}{2} \\ \frac{1}{6} \\ 0 \\ -\frac{1}{30} \\ 0 \\ \frac{1}{42} \\ 0 \end{matrix}$$

$$\sum_j {Q_{ij}}^{-1} = B_i$$

Faulhaber's formula via Pascal −1

$$1^r + 2^r + 3^r + \dots + n^r = U(n,r){Q_r}^{-1}\mathbf{1}$$

where

$$U(n,r) = \frac{1}{r+1}\left(\binom{r+1}{0}n^{r+1}, \dots, (-1)^k\binom{r+1}{k}n^{r+1-k}, \dots, (-1)^r\binom{r+1}{r}n^1\right)$$

and

$$\mathbf{1} = (1, \dots, 1).$$

$$1^3 + 2^3 + 3^3 + 4^3 + 5^3 = U(5,3){Q_3}^{-1}\mathbf{1}$$

$$= \frac{1}{4}\left(\binom{4}{0}5^4, -\binom{4}{1}5^3, \binom{4}{2}5^2, -\binom{4}{3}5^1\right)\begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1/2 & 0 & 0 \\ 2/3 & -5/6 & 1/3 & 0 \\ -1/4 & 3/4 & -3/4 & 1/4 \end{pmatrix}\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$$

$$= \left(\frac{625}{4}, -125, \frac{75}{2}, -5\right) \times \left(1, -\frac{1}{2}, \frac{1}{6}, 0\right)^{-1} = 225.$$

Moessner's Magic: quadratic bootstrapping as a linear transform

In 1951 Alfred Moessner a Bavarian (amateur?) mathematician invented a 'sieve' for listing the kth powers of integers (given a proof shortly afterwards by Oskar Perron, of Perron–Frobenius fame)

Sitzungsberichte

der

mathematisch-naturwissenschaftlichen

Klasse

der

Bayerischen Akademie der Wissenschaften

zu München

Jahrgang 1951

München 1952

Verlag der Bayerischen Akademie der Wissenschaften

In Kommission bei der C. H. Beck'schen Verlagsbuchhandlung München

Eine Bemerkung über die Potenzen der natürlichen Zahlen

Von Alfred Moessner in Gunzenhausen

Vorgelegt von Herrn O. Perron am 2. März 1951

Wenn man aus der Reihe der natürlichen Zahlen jede zweite ausstreicht, so daß nur die Reihe der ungeraden Zahlen übrigbleibt, so besteht deren Summenreihe aus den Quadratzahlen 1, 4, 9, 16, . . .

Diese altbekannte Tatsache läßt eine überraschende Verallgemeinerung zu, die noch nicht bemerkt zu sein scheint:

Wenn man aus der Reihe der natürlichen Zahlen jede dritte ausstreicht und von der übrigbleibenden nun nicht mehr arithmetischen Reihe

1, 2, 4, 5, 7, 8, 10, 11, 13, 14, . . .

die Summenreihe bildet:

1, 3, 7, 12, 19, 27, 37, 48, 61, 75 . . . ,

aus dieser nunmehr jede zweite ausstreicht und von der übrigbleibenden Reihe

1, 7, 19, 37, 61, . . .

wieder die Summenreihe bildet, so kommen die Kubikzahlen 1, 8, 27, 64, 125, . . .

Und allgemein gilt folgendes für jede ganze Zahl $k > 1$:

Wenn man aus der Reihe der natürlichen Zahlen jede k^{te} ausstreicht und von der übrigbleibenden Reihe die Summenreihe bildet, sodann aus dieser jede $(k\text{-}1)^{te}$ Zahl ausstreicht und wieder die Summenreihe bildet, aus dieser sodann jede $(k-2)^{te}$ ausstreicht und abermals die Summenreihe bildet und diesen Prozeß fortsetzt, bis man schließlich beim $(k-1)^{ten}$ Schritt jede zweite Zahl ausstreicht und dann die Summenreihe bildet, so entsteht die Reihe der k^{ten} Potenzen $1^k, 2^k, 3^k, 4^k, \ldots$.

Dieser Satz sei hier zunächst ohne den nicht ganz einfachen Beweis mitgeteilt.

A remark about the powers of the natural numbers

A translation into English of Moessner's 1951 article

If you delete every second number from the series of natural numbers, so that only the series of odd numbers remains, then their sum series consists of the square numbers

1, 4, 9, 16, …

This well-known fact suggests a surprising generalization that seems not to have been noticed yet.

Suppose you delete every third number from the series of numbers, and from the remaining, now no longer arithmetic, series

1, 2, 4, 5, 7, 8, 10, 11, 13, 14, …

take the sum series:

1, 3, 7, 12, 19, 27, 37, 48, 61, …

Now delete every second entry, and from the remaining row

1, 7, 19, 37, 61, …

again take the sum series. Then the cubic numbers appear: 1, 8, 27, 64, 125, …

And in general the following applies to every integer $k > 1$: if you cross out every k-th number from the series of natural numbers and form the sum series from the remaining numbers; then cross out every $(k-1)$-th number from this and form the sum series again, then cross out every $(k-2)$-th number from this and form the sum series again, and continue this process until you finally cross out every second number at the $(k-1)$-th step and then form the sum series, this creates the series of k-th powers 1^k, 2^k, 3^k, 4^k, … .

This theorem is given here first without the not-so-simple proof.

Moessner in matrix form I

If you delete every second number from the series of natural numbers, so that only the series of odd numbers remains, then their sum series consists of the square numbers

1, 4, 9, 16, …

This well-known fact suggests a surprising generalization that seems not to have been noticed yet.

```
sr(7, 2), first_n(7), evalm(sr(7, 2)&*first_n(7))
```

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 9 \\ 16 \end{bmatrix}$$

Suppose you delete every third number from the series of numbers, and from the remaining, now no longer arithmetic, series

1, 2, 4, 5, 7, 8, 10, 11, 13, 14, …

take the sum series:

1, 3, 7, 12, 19, 27, 37, 48, 61, …

```
sr(13, 3), first_n(13), evalm(sr(13, 3)&*first_n(13));
```

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \\ 11 \\ 12 \\ 13 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 7 \\ 12 \\ 19 \\ 27 \\ 37 \\ 48 \\ 61 \end{bmatrix}$$

Moessner in matrix form II

…..

1, 3, 7, 12, 19, 27, 37, 48, 61, …

$sr(13, 3), first\_n(13), evalm(sr(13, 3) \&* first\_n(13));$

$$\begin{bmatrix} 1&0&0&0&0&0&0&0&0&0&0&0&0 \\ 1&1&0&0&0&0&0&0&0&0&0&0&0 \\ 1&1&0&1&0&0&0&0&0&0&0&0&0 \\ 1&1&0&1&1&0&0&0&0&0&0&0&0 \\ 1&1&0&1&1&0&1&0&0&0&0&0&0 \\ 1&1&0&1&1&0&1&1&0&0&0&0&0 \\ 1&1&0&1&1&0&1&1&0&1&0&0&0 \\ 1&1&0&1&1&0&1&1&0&1&1&0&0 \\ 1&1&0&1&1&0&1&1&0&1&1&0&1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \\ 11 \\ 12 \\ 13 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 7 \\ 12 \\ 19 \\ 27 \\ 37 \\ 48 \\ 61 \end{bmatrix}$$

Now delete every second entry, and from the remaining row

1, 7, 19, 37, 61, …

again take the sum series. Then the cubic numbers appear: 1, 8, 27, 64, 125, …

$sr(9, 2), sr(13, 3), first\_n(13), evalm(sr(9, 2) \&* sr(13, 3) \&* first\_n(13));$

$$\begin{bmatrix} 1&0&0&0&0&0&0&0&0 \\ 1&0&1&0&0&0&0&0&0 \\ 1&0&1&0&1&0&0&0&0 \\ 1&0&1&0&1&0&1&0&0 \\ 1&0&1&0&1&0&1&0&1 \end{bmatrix}, \begin{bmatrix} 1&0&0&0&0&0&0&0&0&0&0&0&0 \\ 1&1&0&0&0&0&0&0&0&0&0&0&0 \\ 1&1&0&1&0&0&0&0&0&0&0&0&0 \\ 1&1&0&1&1&0&0&0&0&0&0&0&0 \\ 1&1&0&1&1&0&1&0&0&0&0&0&0 \\ 1&1&0&1&1&0&1&1&0&0&0&0&0 \\ 1&1&0&1&1&0&1&1&0&1&0&0&0 \\ 1&1&0&1&1&0&1&1&0&1&1&0&0 \\ 1&1&0&1&1&0&1&1&0&1&1&0&1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \\ 11 \\ 12 \\ 13 \end{bmatrix}, \begin{bmatrix} 1 \\ 8 \\ 27 \\ 64 \\ 125 \end{bmatrix}$$

Moessner in matrix form III

…..

Now delete every second entry, and from the remaining row

1, 7, 19, 37, 61, …

again take the sum series. Then the cubic numbers appear: 1, 8, 27, 64, 125, …

```
sr(9, 2), sr(13, 3), first_n(13), evalm(sr(9, 2)&* sr(13, 3)&*first_n(13));
```

$$\begin{bmatrix} 1&0&0&0&0&0&0&0&0\\ 1&0&1&0&0&0&0&0&0\\ 1&0&1&0&1&0&0&0&0\\ 1&0&1&0&1&0&1&0&0\\ 1&0&1&0&1&0&1&0&1 \end{bmatrix}, \begin{bmatrix} 1&0&0&0&0&0&0&0&0&0&0&0&0\\ 1&1&0&0&0&0&0&0&0&0&0&0&0\\ 1&1&0&1&0&0&0&0&0&0&0&0&0\\ 1&1&0&1&1&0&0&0&0&0&0&0&0\\ 1&1&0&1&1&0&1&0&0&0&0&0&0\\ 1&1&0&1&1&0&1&1&0&0&0&0&0\\ 1&1&0&1&1&0&1&1&0&1&0&0&0\\ 1&1&0&1&1&0&1&1&0&1&1&0&0\\ 1&1&0&1&1&0&1&1&0&1&1&0&1 \end{bmatrix}, \begin{bmatrix} 1\\2\\3\\4\\5\\6\\7\\8\\9\\10\\11\\12\\13 \end{bmatrix}, \begin{bmatrix} 1\\8\\27\\64\\125 \end{bmatrix}$$

And what is the linear transform producing the cubes?

```
evalm(sr(9, 2)&* sr(13, 3))
```

$$\begin{bmatrix} 1&0&0&0&0&0&0&0&0&0&0&0&0\\ 2&1&0&1&0&0&0&0&0&0&0&0&0\\ 3&2&0&2&1&0&1&0&0&0&0&0&0\\ 4&3&0&3&2&0&2&1&0&1&0&0&0\\ 5&4&0&4&3&0&3&2&0&2&1&0&1 \end{bmatrix}$$

Moessner in matrix form IV

And what is the linear transform producing the cubes?

$evalm(sr(9,2) \&^* sr(13,3))$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & 2 & 0 & 2 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 4 & 3 & 0 & 3 & 2 & 0 & 2 & 1 & 0 & 1 & 0 & 0 & 0 \\ 5 & 4 & 0 & 4 & 3 & 0 & 3 & 2 & 0 & 2 & 1 & 0 & 1 \end{bmatrix}$$

What if this is is a repeated, down-shifted block?

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & 2 & 0 & 2 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 4 & 3 & 0 & 3 & 2 & 0 & 2 & 1 & 0 & 1 & 0 & 0 & 0 \\ 5 & 4 & 0 & 4 & 3 & 0 & 3 & 2 & 0 & 2 & 1 & 0 & 1 \end{bmatrix}$$

$\cdots$

$$\rightarrow \quad \begin{array}{ccc} 1^1 \times 0^0 & 1^0 \times 0^1 & 0 \\ 2^1 \times 1^0 & 2^0 \times 1^1 & 0 \\ 3^1 \times 2^0 & 3^0 \times 2^1 & 0 \\ 4^1 \times 3^0 & 4^0 \times 3^1 & 0 \\ 5^1 \times 4^0 & 5^0 \times 4^1 & 0 \\ 6^1 \times 5^0 & 6^0 \times 5^1 & 0 \\ & \cdots & \end{array}$$

Moessner in matrix form V

What is involved in listing the 5th powers?

Omit every 5th number

$$\mathrm{sr}(10,5) = \begin{bmatrix} 1&0&0&0&0&0&0&0&0&0\\ 1&1&0&0&0&0&0&0&0&0\\ 1&1&1&0&0&0&0&0&0&0\\ 1&1&1&1&0&0&0&0&0&0\\ 1&1&1&1&0&1&0&0&0&0\\ 1&1&1&1&0&1&1&0&0&0\\ 1&1&1&1&0&1&1&1&0&0\\ 1&1&1&1&0&1&1&1&1&0 \end{bmatrix}$$

Omit every 4th number

$$\mathrm{sr}(10,4) = \begin{bmatrix} 1&0&0&0&0&0&0&0&0&0\\ 1&1&0&0&0&0&0&0&0&0\\ 1&1&1&0&0&0&0&0&0&0\\ 1&1&1&0&1&0&0&0&0&0\\ 1&1&1&0&1&1&0&0&0&0\\ 1&1&1&0&1&1&1&0&0&0\\ 1&1&1&0&1&1&1&0&1&0\\ 1&1&1&0&1&1&1&0&1&1 \end{bmatrix}$$

Omit every 3rd number

$$\mathrm{sr}(10,3) = \begin{bmatrix} 1&0&0&0&0&0&0&0&0&0\\ 1&1&0&0&0&0&0&0&0&0\\ 1&1&0&1&0&0&0&0&0&0\\ 1&1&0&1&1&0&0&0&0&0\\ 1&1&0&1&1&0&1&0&0&0\\ 1&1&0&1&1&0&1&1&0&0\\ 1&1&0&1&1&0&1&1&0&1 \end{bmatrix}$$

Omit every 2nd number

$$\mathrm{sr}(10,2) = \begin{bmatrix} 1&0&0&0&0&0&0&0&0&0\\ 1&0&1&0&0&0&0&0&0&0\\ 1&0&1&0&1&0&0&0&0&0\\ 1&0&1&0&1&0&1&0&0&0\\ 1&0&1&0&1&0&1&0&1&0 \end{bmatrix}$$

These matrices represent Moessner's Summenreihe, his sum series.

The number of rows reduces since we omit the row that says 'don't add the k-th natural number'.

$$\begin{bmatrix} 1&0&0&0&0&0&0&0&0&0\\ 1&1&0&0&0&0&0&0&0&0\\ 1&1&0&1&0&0&0&0&0&0\\ 1&1&0&1&1&0&0&0&0&0\\ 1&1&0&1&1&0&1&0&0&0\\ 1&1&0&1&1&0&1&1&0&0\\ 1&1&0&1&1&0&1&1&0&1 \end{bmatrix}$$

- ← Take 1st number
- ← Add 2nd number
- ← Add 4th number
- ← Add 5th number
- ← Add 7th number
- …

sr (n,k) has roughly

$$\frac{k-1}{k}\times n$$

rows. In fact, precisely

$$\left\lceil \frac{k-1}{k}n \right\rceil .$$

Moessner in matrix form VI

Suppose we want to list the first six 5th powers.

The product of sum series matrices will be

$$\mathrm{sr}(n_2,2)\times \mathrm{sr}(n_3,3)\times \mathrm{sr}(n_4,4)\times \mathrm{sr}(n_5,5)\times \begin{bmatrix}1\\2\\3\\\vdots\end{bmatrix}$$

with the n_i chosen to produce a product matrix of six rows.

So n_2 is 12 to give $\left\lceil \frac{1}{2}\times 12 \right\rceil = 6$

So then n_3 is 18 to give $\left\lceil \frac{2}{3}\times 18 \right\rceil = 12$

So then n_4 is 24 to give $\left\lceil \frac{3}{4}\times 24 \right\rceil = 18$

So finally n_5 is 30 to give $\left\lceil \frac{4}{5}\times 30 \right\rceil = 24$

$$[6\times 12] \times [12\times 18] \times [18\times 24] \times [24\times 30] \times [30\times 1]$$

Moessner in matrix form VII

And here is the calculation

$$\mathrm{sr}(n_2,2)\times \mathrm{sr}(n_3,3)\times \mathrm{sr}(n_4,4)\times \mathrm{sr}(n_5,5)\times \begin{bmatrix}1\\2\\3\\ \vdots\end{bmatrix}$$

$$6\times 12 \;\times\; 12\times 18 \;\times\; 18\times 24 \;\times\; 24\times 30 \;\times\; \begin{matrix}30\\ \times\\ 1\end{matrix}$$

```
s := evalm(sr(12, 2)&*sr(18, 3)&*sr(24, 4)&*sr(30, 5))
```

$$s := \left[\begin{array}{cccccccccccccccccccccccccccccc}
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
8&4&2&1&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
27&18&12&8&0&8&4&2&1&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
64&48&36&27&0&27&18&12&8&0&8&4&2&1&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
125&100&80&64&0&64&48&36&27&0&27&18&12&8&0&8&4&2&1&0&1&0&0&0&0&0&0&0&0&0\\
216&180&150&125&0&125&100&80&64&0&64&48&36&27&0&27&18&12&8&0&8&4&2&1&0&1&0&0&0&0
\end{array}\right]$$

```
evalm(s&*first_n(30))
```

$$\begin{bmatrix}1\\32\\243\\1024\\3125\\7776\end{bmatrix}$$

Moessner in matrix form VIII

And the Moessner matrix?

$$
\mathrm{sr}(n_2,2)\times\mathrm{sr}(n_3,3)\times\mathrm{sr}(n_4,4)\times\mathrm{sr}(n_5,5)=
\left[\begin{array}{ccccc|ccccc|ccccc|ccccc|ccccc|ccccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
8 & 4 & 2 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
27 & 18 & 12 & 8 & 0 & 8 & 4 & 2 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
64 & 48 & 36 & 27 & 0 & 27 & 18 & 12 & 8 & 0 & 8 & 4 & 2 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
125 & 100 & 80 & 64 & 0 & 64 & 48 & 36 & 27 & 0 & 27 & 18 & 12 & 8 & 0 & 8 & 4 & 2 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
216 & 180 & 150 & 125 & 0 & 125 & 100 & 80 & 64 & 0 & 64 & 48 & 36 & 27 & 0 & 27 & 18 & 12 & 8 & 0 & 8 & 4 & 2 & 1 & 0 & 1 & 0 & 0 & 0 & 0
\end{array}\right]
$$

$$
\begin{array}{ccccc}
1^3\times 0^0 & 1^2\times 0^1 & 1^1\times 0^2 & 1^0\times 0^3 & 0 \\
2^3\times 1^0 & 2^2\times 1^1 & 2^1\times 1^2 & 2^0\times 1^3 & 0 \\
3^3\times 2^0 & 3^2\times 2^1 & 3^1\times 2^2 & 3^0\times 2^3 & 0 \\
4^3\times 3^0 & 4^2\times 3^1 & 4^1\times 3^2 & 4^0\times 3^3 & 0 \\
5^3\times 4^0 & 5^2\times 4^1 & 5^1\times 4^2 & 5^0\times 4^3 & 0 \\
6^3\times 5^0 & 6^2\times 5^1 & 6^1\times 5^2 & 6^0\times 5^3 & 0 \\
 & & \cdots & &
\end{array}
$$

Moessner in matrix form IX

The Moessner matrix block is shifted down by 1 each time.

So the result of multiplying it into the column of natural numbers is the same as multiplying across and up the rows of the block:

1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
8	4	2	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
27	18	12	8	0	8	4	2	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
64	48	36	27	0	27	18	12	8	0	8	4	2	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
125	100	80	64	0	64	48	36	27	0	27	18	12	8	0	8	4	2	1	0	1	0	0	0	0	0	0	0	0	0
216	180	150	125	0	125	100	80	64	0	64	48	36	27	0	27	18	12	8	0	8	4	2	1	0	1	0	0	0	0

$1^3\times 0^0$	$1^2\times 0^1$	$1^1\times 0^2$	$1^0\times 0^3$	0
11	12	13	14	15
$2^3\times 1^0$	$2^2\times 1^1$	$2^1\times 1^2$	$2^0\times 1^3$	0
6	7	8	9	10
$3^3\times 2^0$	$3^2\times 2^1$	$3^1\times 2^2$	$3^0\times 2^3$	0
1	2	3	4	5
$4^3\times 3^0$	$4^2\times 3^1$	$4^1\times 3^2$	$4^0\times 3^3$	0
$5^3\times 4^0$	$5^2\times 4^1$	$5^1\times 4^2$	$5^0\times 4^3$	0
$6^3\times 5^0$	$6^2\times 5^1$	$6^1\times 5^2$	$6^0\times 5^3$	0
		…		

15 → 11

10 → 101

5 → 131

$243 = 3^5$

This gives a formula for kth powers which is a bit cumbersome but fits our idea of quadratic bootstrapping:

$$n^k = \sum_{r=1}^{n}\sum_{c=0}^{k-2}((n-r)k+c+1)(r-1)^c r^{k-2-c}$$

“This theorem is given here first without the not-so-simple proof.”