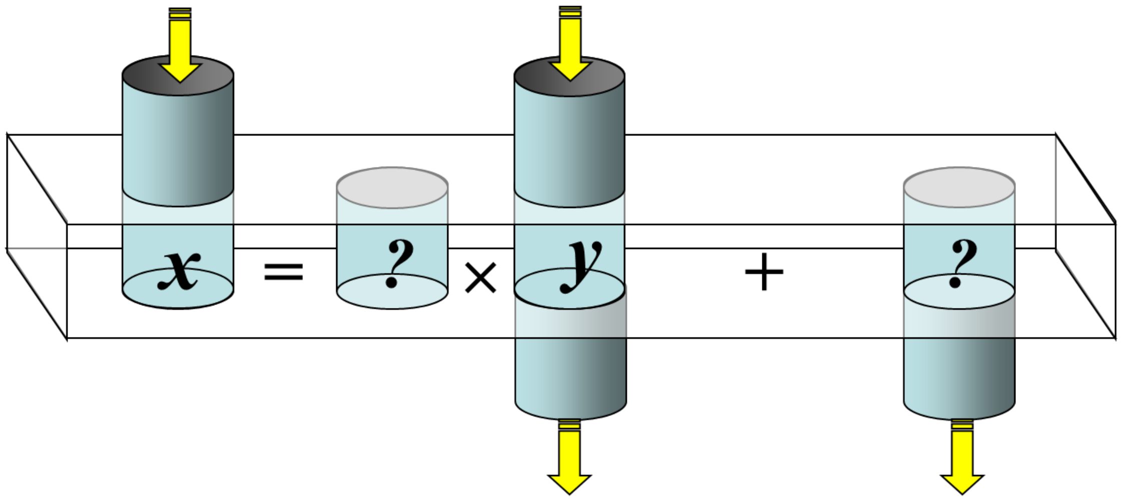
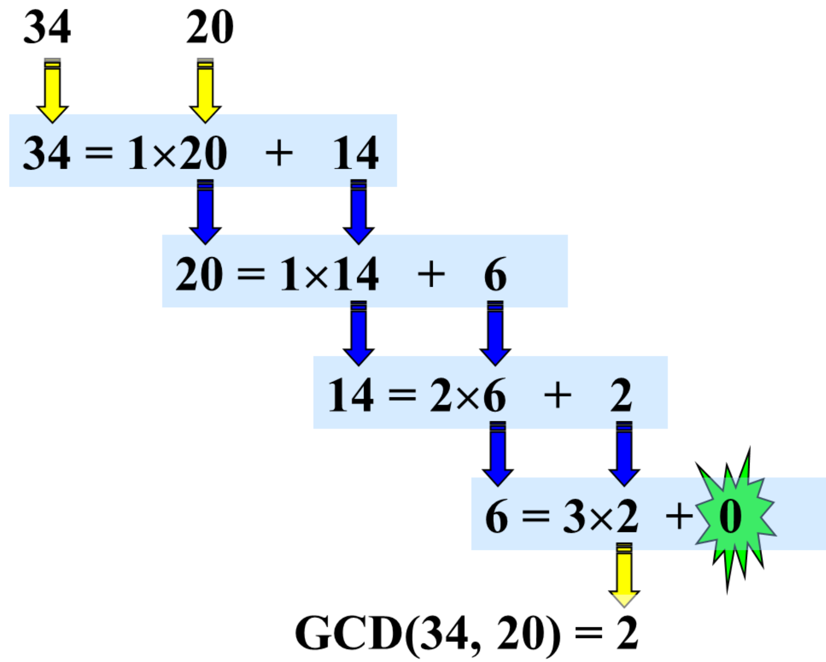
Lamé's Theorem *Suppose the greatest common divisor of x and y, $x > y \geq 1$, is computed, using Euclid's algorithm, in n steps. Then $n \leq \log_\varphi(x\sqrt{5})$, where the base of the logarithm is $\varphi = (1+\sqrt{5})/2$, the golden ratio. The value of n is maximised when y and x are consecutive Fibonacci numbers, F_{n-1} and F_n.*



Bolt-together for Euclid's Algorithm



Euclid's algorithm applies the 'gadget', above-left, until the remainder, the right-hand output, becomes zero. The greatest common divisor (GCD) is then the left-hand output. Try replacing input 20, in the example, with 21. You will find that Euclid's algorithm generates remainders which, from 34 and 21, continue back down the Fibonacci sequence: 13, 8, 5, 3, 2, 1, before ending in zero. In general, if we start with $x = x_0$ and $y = x_1$ and Euclid's algorithm terminates in n steps, then we get a sequence $x = x_0, \ldots, x_{n+1} = 0$. We know that x_n, the GCD, satisfies $x_n \geq 1 = F_1$. Then $x_{n-1} \geq x_n \geq 1 = F_2$; $x_{n-2} \geq x_{n-1} + x_n \geq 1 + 1 = 2 = F_3$; $x_{n-3} \geq x_{n-2} + x_{n-1} \geq 1 + 2 = 3 = F_4$; ..., and the Fibonacci sequence magically appears as a sequence of upper bounds. Now $x = x_0 = x_{n-n} \geq F_{n+1} \approx \frac{1}{\sqrt{5}}\varphi^{n+1}$ by Binet's Formula and Lamé's Theorem follows by taking logs of both sides.

The Fibonacci sequence dates back to the thirteenth century with the work of Leonardo of Pisa (son of Bonacci, the 'amiable one'); Euclid's algorithm appears in book VII of the *Elements*, appearing around three hundred BC; the golden ratio *may* have been used by the ancient Egyptians in three *thousand* BC. Gabriel Lamé's theorem of 1844 perhaps draws on five millennia of mathematical history!

Web link: www.theoremoftheday.org/Docs/BachZeleny_notes.pdf (notes by Eric Bach and Dalibor Zelený). Markowsky is essential reading on the historical and cultural significance of the golden ratio (www.cs.umaine.edu/~markov/GoldenRatio.pdf, 2MB).

Further reading: *The Art of Computer Programming* by Donald E. Knuth, Addison-Wesley, 1999 (this edition). Vol. 2, section 4.