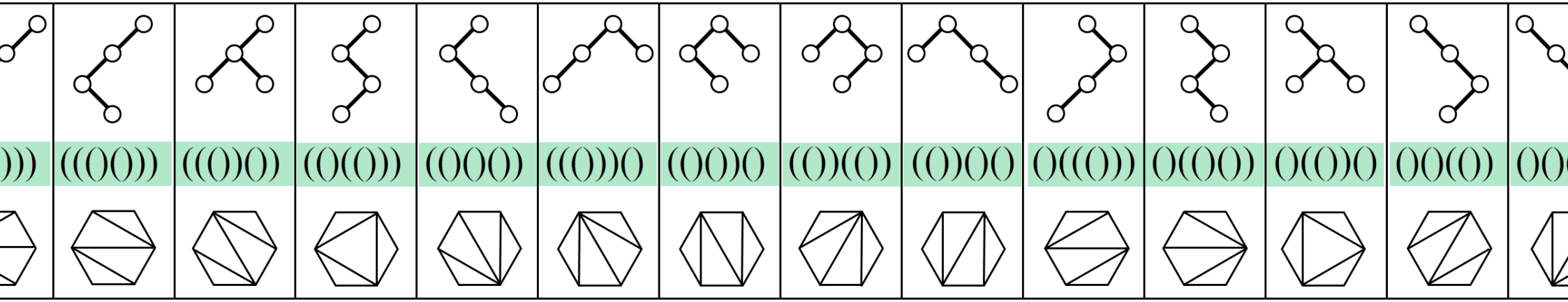
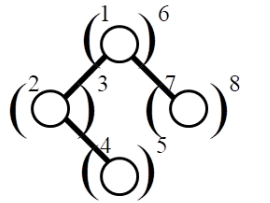
THEOREM OF THE DAY

The Catalan–Euler–Segner Bijection *Binary trees on n nodes are equinumerous with triangulations of a convex polygon on n + 2 vertices, this number being the n-th Catalan number* $C_n = \frac{1}{n+1}\binom{2n}{n}$.



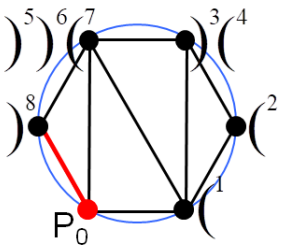
A binary tree is either the empty tree or consists of a *root node* joined to left and right *subtrees*, each of which is again a binary tree. By convention, it is depicted as branching downwards with each node branching left and right. Empty subtrees are omitted from the drawing. There are fourteen binary trees on four nodes as shown in the table above (slightly cropped, but the incomplete trees at each end are not hard to reconstruct). The last row of the table shows the fourteen triangulations of the hexagon (i.e. three non-crossing chords which divide the interior of the hexagon into four triangles).

We want to construct, from each binary tree, a distinct triangulation, and vice versa. This offers a bijection between the top and bottom rows of the table. Our construction will pass via sequences of four pairs of brackets which are valid, meaning we never have more closed pairs than open pairs: (()))(is not valid, for example. There are fourteen of these too.



1 2 3 4 5 6 7 8

((()())())



The algorithm on the right traverses a binary tree T, as illustrated on the left, descending recursively through its subtrees appending brackets to a valid sequence as it goes.

```
Tree_to_brackets(T)
  if T empty then return
  write "("
  Tree_to_brackets(left subtree of T)
  write ")"
  Tree_to_brackets(right subtree of T)
```

Now, take a valid bracket sequence B made of n pairs of brackets. Draw a circle and on it place $n + 2$ vertices. Start by adding an edge to the bottom left vertex P_0 from its anticlockwise predecessor on the circle. Run the algorithm on the right starting at next(P_0), the anticlockwise successor of P_0. In the algorithm, 'valid edge', should join to that most recently visited vertex (of $P_0, P_1, ...$) which avoids crossing through existing edges.

Et voilà! (The correctness and reversibility of these constructions, however, is left as an exercise!)

```
Brackets_to_polygon(B)
  v := next(P0)
  do
    do
      b := next(B)
      add new valid edge from v
    while b ≠ '(' and b ≠ Blast
    v := next(P)
  while v ≠ P0
```

Leonhard Euler and Johann Segner successfully enumerated polygon triangulations in the mid-eighteen century. In the 1830s, Eugène Catalan found the formula for C_n as given above and studied bracket sequences and the closely related ballot numbers. The term 'Catalan number' has been traced by Igor Pak to John Riordan's 1968 monograph *Combinatorial Identities*.

Web links: www.math.ucla.edu/~pak/lectures/Cat/pakcat.htm

Further reading: *Catalan Numbers with Applications* by Thomas Koshy, Oxford University Press, 2008.