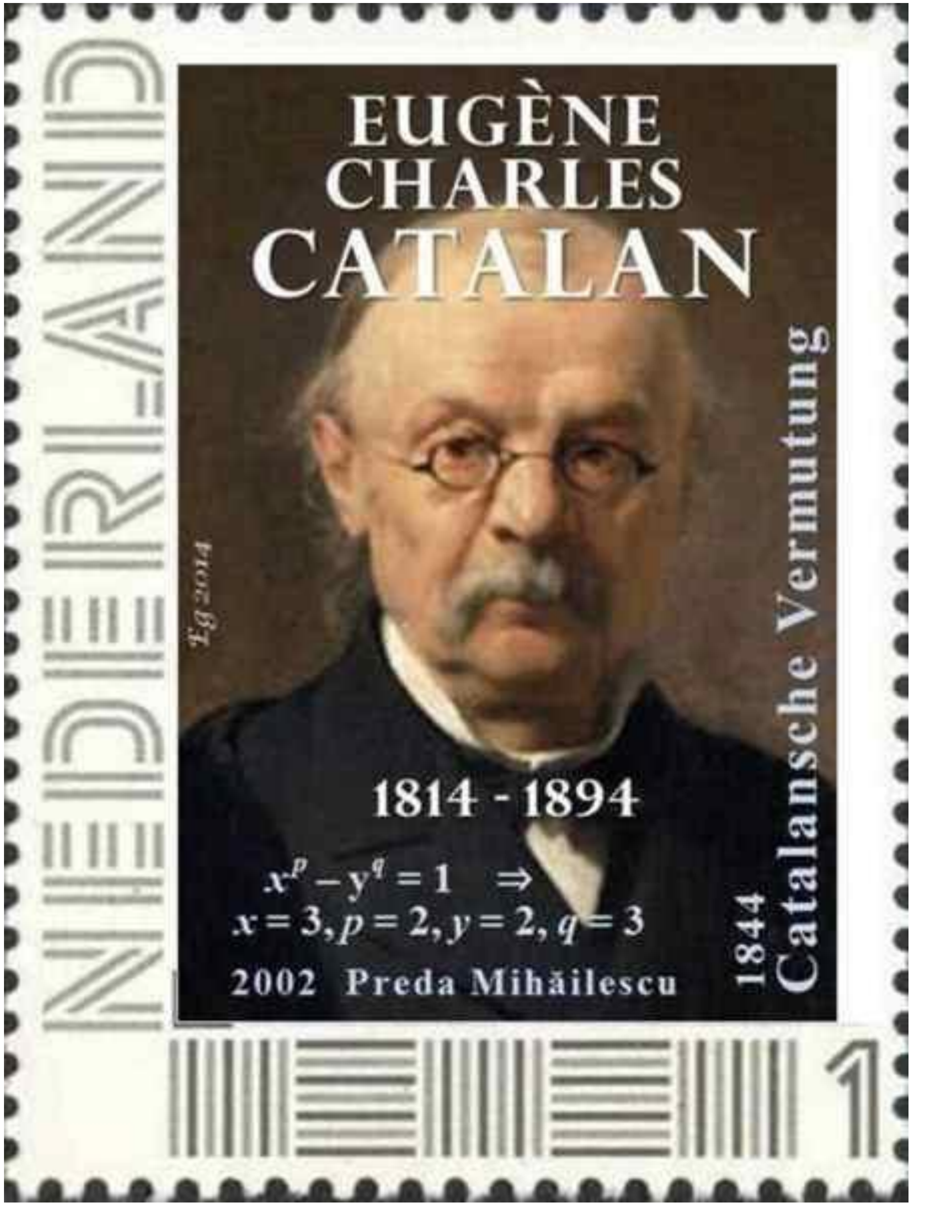
The Catalan Numbers

Robin Whitty
Maths Study Group,
April 2025



A000108

"This is probably the longest entry in the OEIS, and rightly so."

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THE ON-LINE ENCYCLOPEDIA OF INTEGER SEQUENCES ®

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catalan numbers [Search] Hints

(Greetings from The On-Line Encyclopedia of Integer Sequences!)

Search: **catalan numbers**

Displaying 1-10 of 3505 results found. page 1 2 3 4 5 6 7 8 9 10 ... 351

Sort: relevance | references | number | modified | created Format: long | short | data

A000108 **Catalan numbers**: C(n) = binomial(2n,n)/(n+1) = (2n)!/(n!(n+1)!). +40
(Formerly M1459 N0577) 4153

1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796, 58786, 208012, 742900, 2674440, 9694845, 35357670, 129644790, 477638700, 1767263190, 6564120420, 24466267020, 91482563640, 343059613650, 1289904147324, 4861946401452, 18367353072152, 69533550916004, 263747951750360, 1002242216651368, 3814986502092304

(list; graph; refs; listen; history; text; internal format)

OFFSET 0,3

COMMENTS

These were formerly sometimes called Segner **numbers**.

A very large **number** of combinatorial interpretations are known - see references, esp. R. P. Stanley, "**Catalan Numbers**", Cambridge University Press, 2015. This is probably the longest entry in the OEIS, and rightly so.

The solution to Schröder's first problem: **number** of ways to insert n pairs of parentheses in a word of n+1 letters. E.g., for n=2 there are 2 ways: ((ab)c) or (a(bc)); for n=3 there are 5 ways: ((ab)(cd)), (((ab)c)d), ((a(bc))d), (a((bc)d)), (a(b(cd))).

Consider all the binomial(2n,n) paths on squared paper that (i) start at (0, 0), (ii) end at (2n, 0) and (iii) at each step, either make a (+1,+1) step or a (+1,-1) step. Then the **number** of such paths that never go below the x-axis (Dyck paths) is C(n). [Chung-Feller]

Number of noncrossing partitions of the n-set. For example, of the 15 set partitions of the 4-set, only [{13},{24}] is crossing, so there are a(4)=14 noncrossing partitions of 4 elements. - Joerg Arndt, Jul 11 2011

Noncrossing partitions are partitions of genus 0. - Robert Coquereaux, Feb 13 2024

a(n-1) is the **number** of ways of expressing an n-cycle (123...n) in the symmetric group S_n as a product of

Igor Pak's Catalan page

math.ucla.edu/~pak/lectures/Cat/pakcat.htm

Catalan Numbers Page

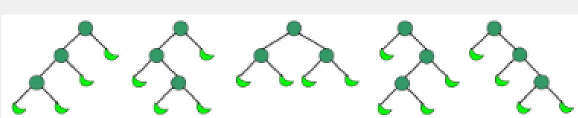
Content: Below is a list of articles on a diverse topics related to Catalan numbers and their generalizations. I emphasized historically significant works, as well as some *bijective*, *geometric* and *probabilistic* results.

Warning: This list is vastly incomplete as I included only downloadable articles and books (sometimes, by subscription) that I found useful at different times. I do plan to gradually expand it, but will try not to overwhelm the list, so many related results can be obtained by forward and backward reference searches. Let me know if you find it useful.

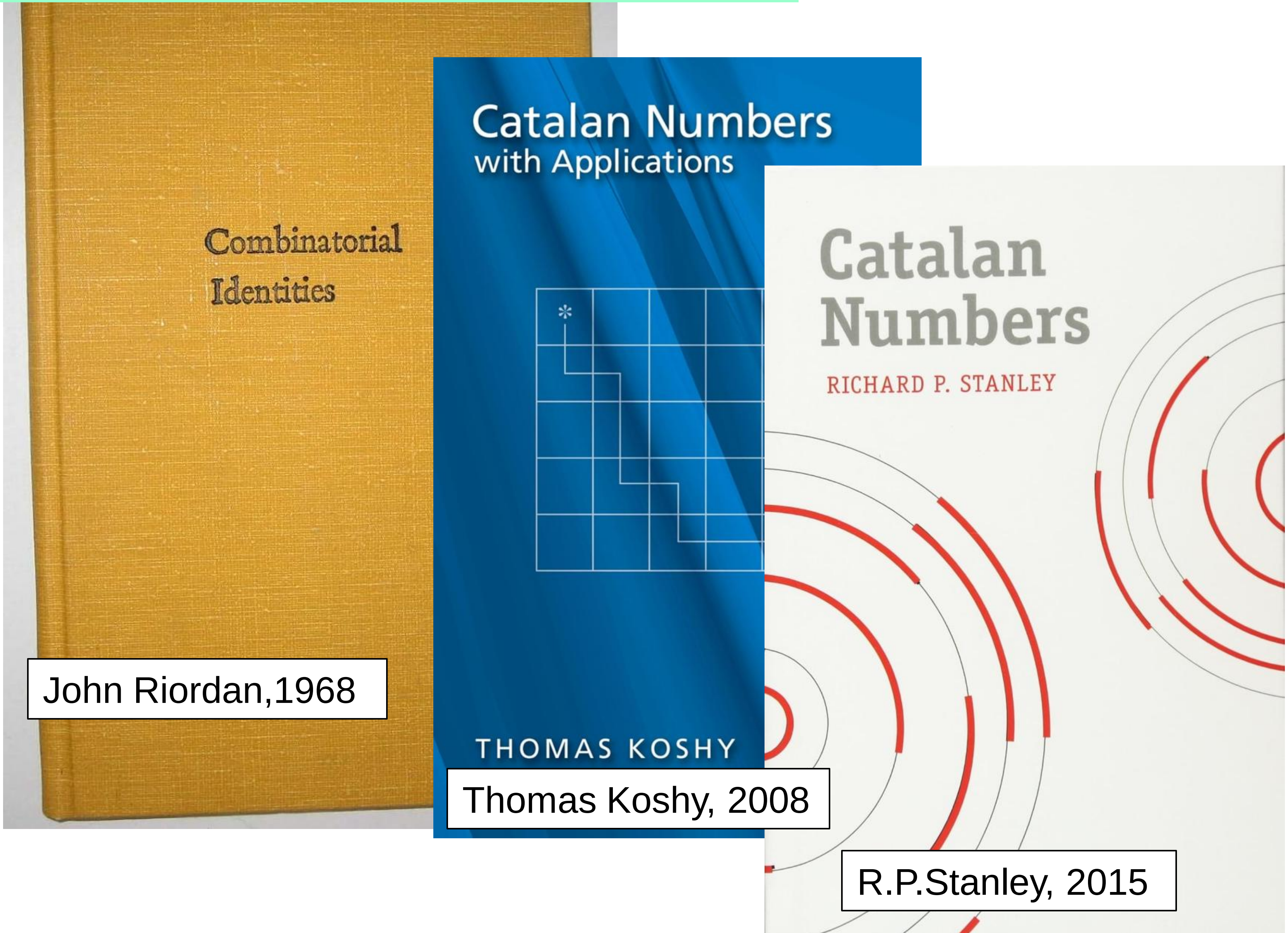
Basics:

$C_n = \frac{1}{n+1}\binom{2n}{n} = \binom{2n}{n} - \binom{2n}{n+1}$ for all $n \geq 0$.

Larger values: at OEIS. Examples and Images: Catalan numbers (MacTutor History of Math.) Another meaning.



In a bookshop near you…

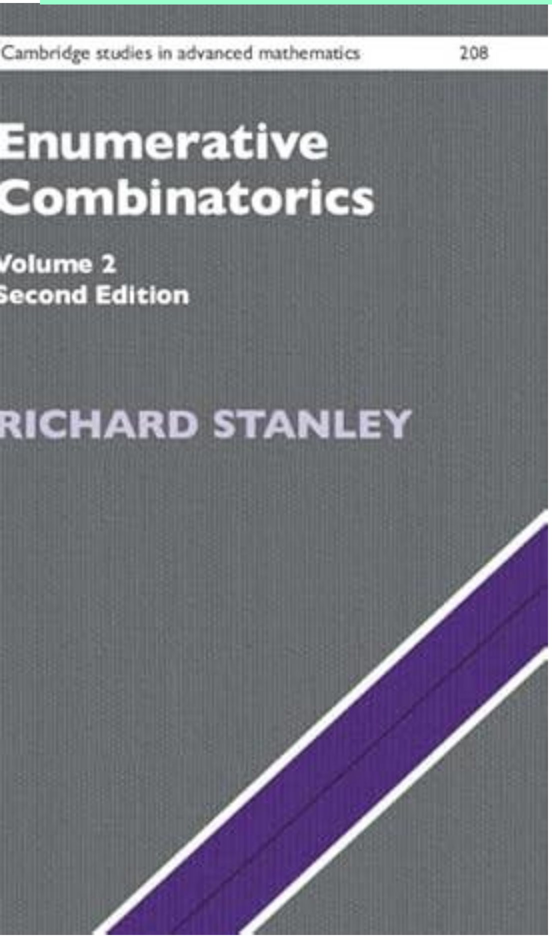


John Riordan,1968

Thomas Koshy, 2008

R.P.Stanley, 2015

The famous Exercise 6.19



Exercises on Catalan and Related Numbers

excerpted from *Enumerative Combinatorics*, vol. 2

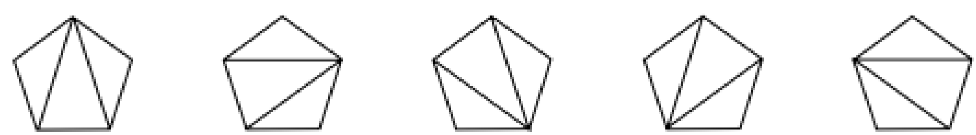
(published by Cambridge University Press 1999)

by Richard P. Stanley

version of 23 June 1998

19. [1]–[3+] Show that the Catalan numbers $C_n = \frac{1}{n+1}\binom{2n}{n}$ count the number of elements of the 66 sets S_i, (a) $\leq i \leq$ (nnn) given below. We illustrate the elements of each S_i for $n = 3$, hoping that these illustrations will make any undefined terminology clear. (The terms used in (vv)–(yy) are defined in Chapter 7.) Ideally S_i and S_j should be proved to have the same cardinality by exhibiting a simple, elegant bijection $\phi_{ij} : S_i \to S_j$ (so 4290 bijections in all). In some cases the sets S_i and S_j will actually coincide, but their descriptions will differ.

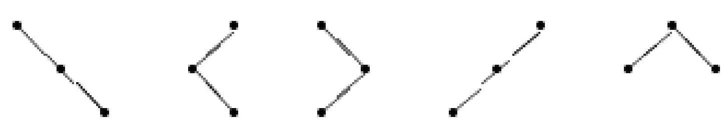
(a) triangulations of a convex $(n+2)$-gon into n triangles by $n-1$ diagonals that do not intersect in their interiors



(b) binary parenthesizations of a string of $n+1$ letters

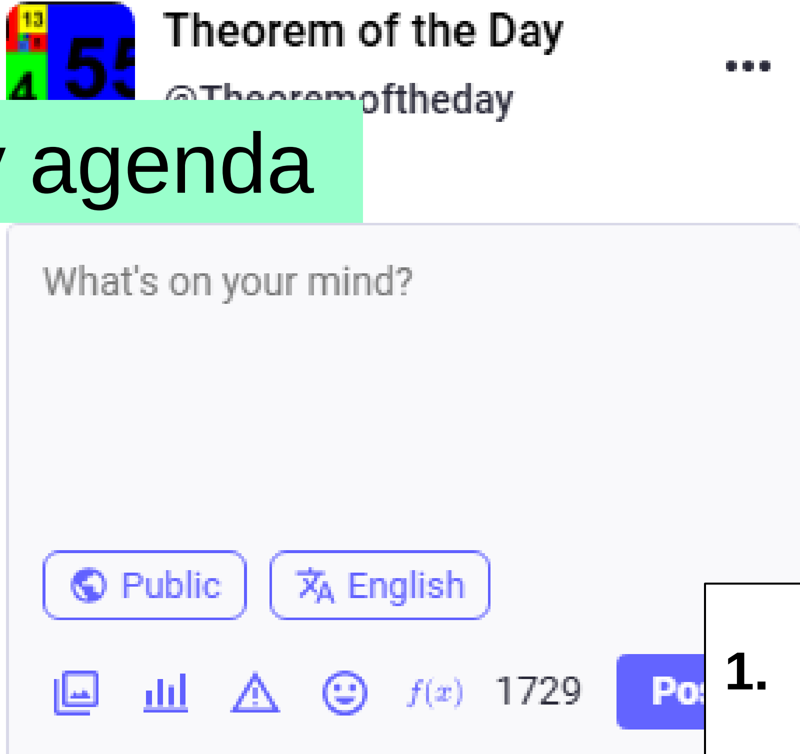
$(xx \cdot x)x \quad x(xx \cdot x) \quad (x \cdot xx)x \quad x(x \cdot xx) \quad xx \cdot xx$

(c) binary trees with n vertices



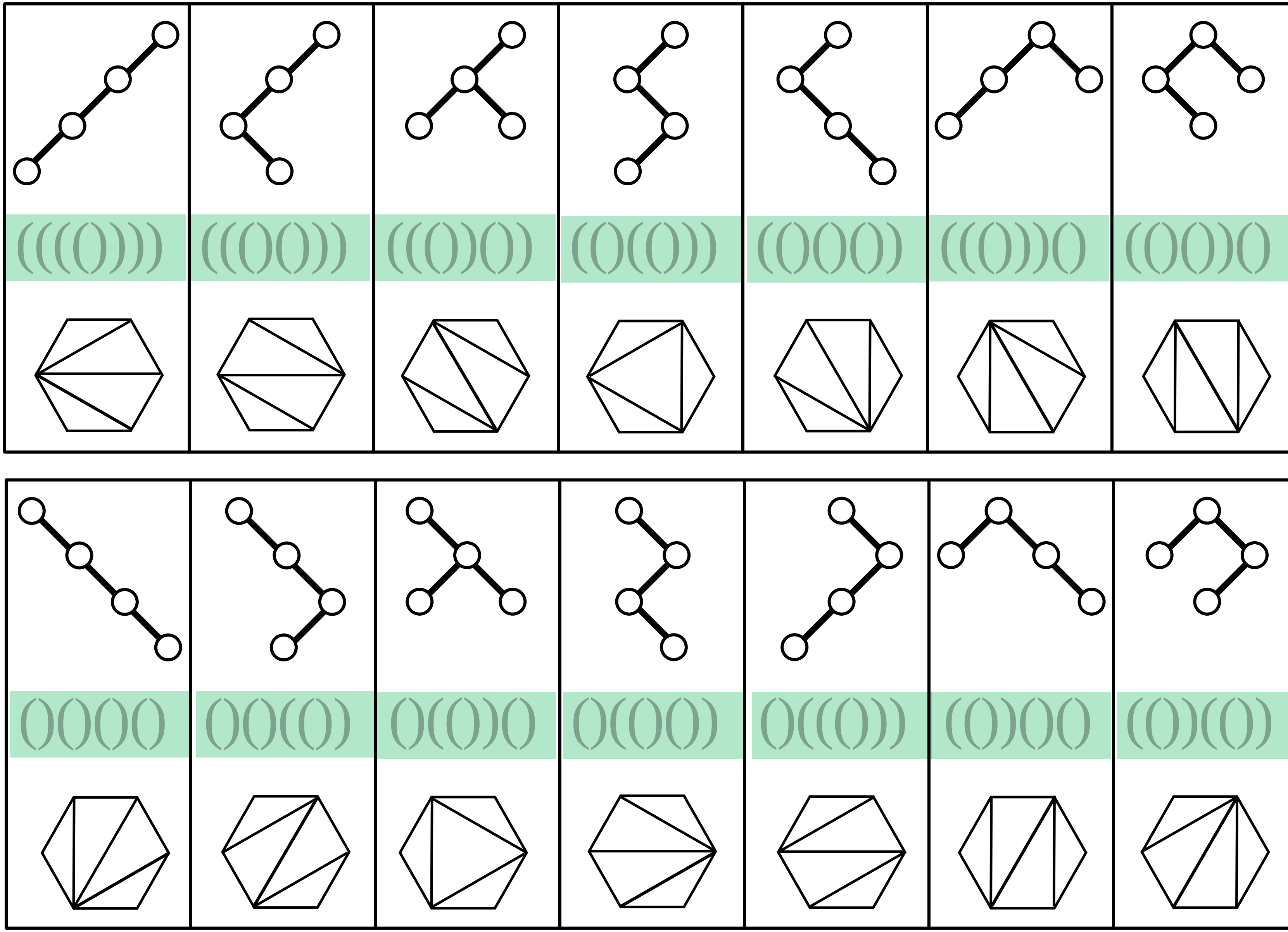
More: math.mit.edu/~rstan/

My agenda

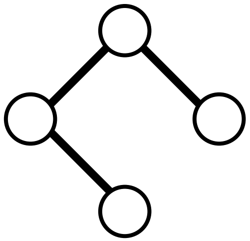


1. **Catalan bijections**
2. **Voting sequences**
3. **Catalan's triangle**
4. **The Catalan triangle formula**
5. **Catalan numbers again**

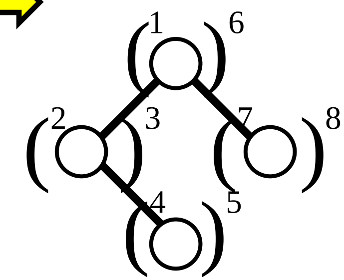
Catalan bijections 1



Catalan bijections 2

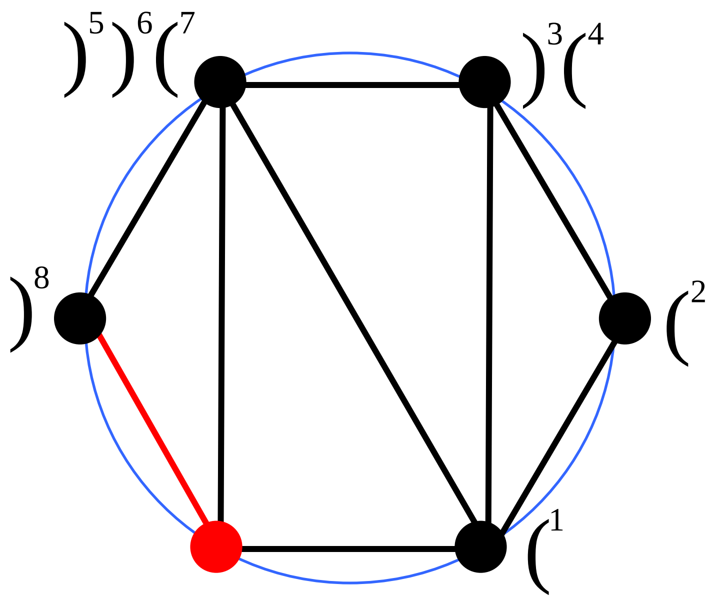


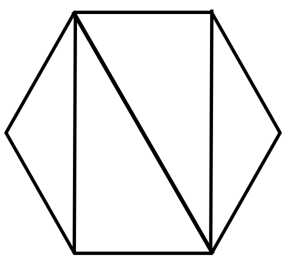
```
Tree_to_brackets(T)
   if T empty then return
   write "("
   Tree_to_brackets(left subtree of T)
   write ")"
   Tree_to_brackets(right subtree of T)
end
```



1 2 3 4 5 6 7 8

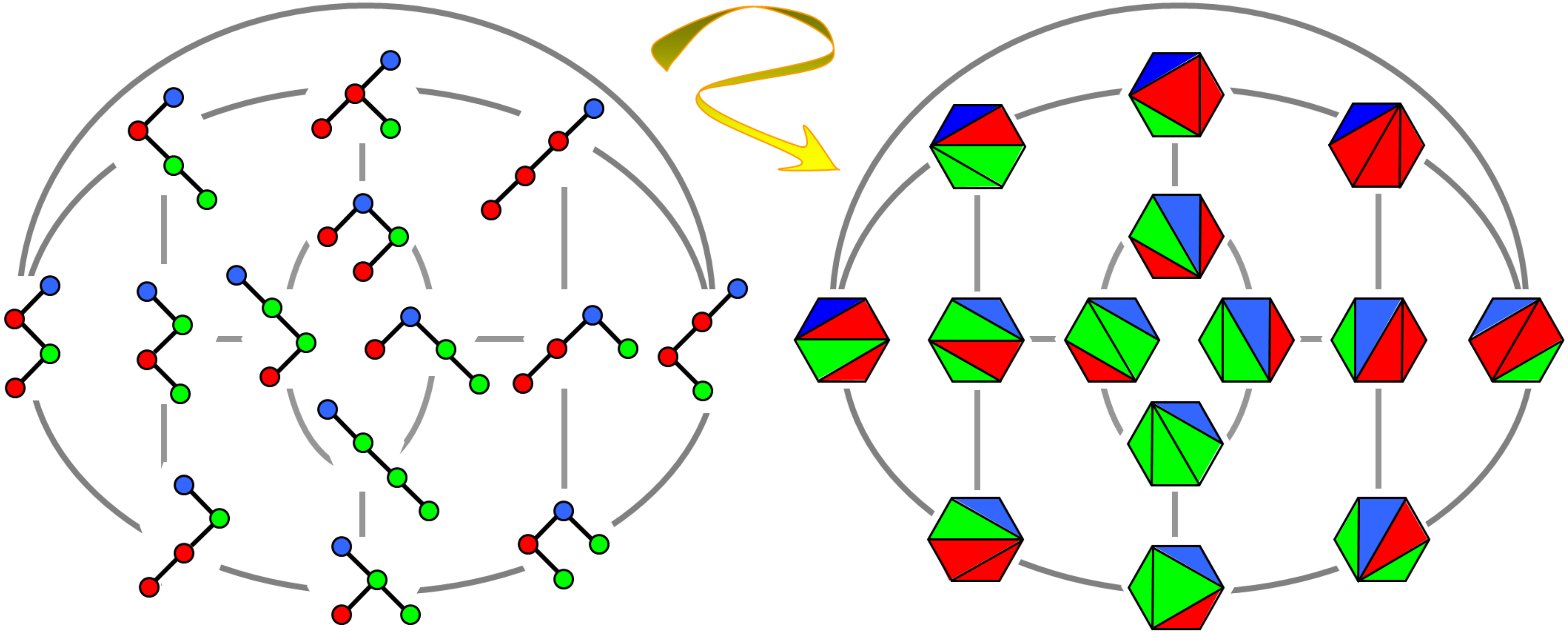
(()())()





Catalan bijections 3

Walking through the binary trees by ‘rotation’. In 1988, Daniel Sleator, Robert Tarjan and William Thurston used a bijection to polygon triangulations (plus a switch to 3D plus a switch to hyperbolic geometry!) to bound ‘rotation distance’ between two trees



More: theoremoftheday.org/Theorems.html#192

Voting sequences 1

Eugène Catalan was maybe the first (1838) to count bracket sequences = *n* pairs of brackets permuted so as to never have more closed brackets than open:

(( ) ( ) ( )) ✓

(( ))) (( ) ✗

He was also the first (1839) to define ballot numbers which he analysed in terms of Catalan numbers, but disguised as triangulations.

X X Y X Y X Y Y ✓ *Y*s never exceed *X*s

X X Y Y Y X X Y ✗ *Y*s sometimes exceed *X*s

X X Y X Y X Y Y X ✓ *X* maintains a winning margin over *Y*

Voting sequences 2

Ballot Theorem
2 candidates X and Y
N voters
X wins by a margin of W votes.
Probability that, from first vote, X is always ahead is W/N.

More: theoremoftheday.org/Theorems.html#252

Aim was to find a ‘natural’ proof of this.
Means finding a ‘natural’ way to count ‘good’ sequences of Xs and Ys.

1	X	X	Y	X	Y	X
2	X	X	X	Y	Y	X
3	X	X	Y	X	X	Y
4	X	X	X	Y	X	Y
5	X	X	X	X	Y	Y.

5/15 sequences for $N = 6$, $W = 2$ are good.

Voting sequences 3

1	X	X	Y	X	Y	X
2	X	X	X	Y	Y	X
3	X	X	Y	X	X	Y
4	X	X	X	Y	X	Y
5	X	X	X	X	Y	Y.

5/15 sequences for $N = 6$, $W = 2$ are good.

An obvious continuation is now to find a 'base' voting sequence from which we can generate all 'good' sequences and no 'bad' sequences, ensuring that no double-counting occurs. And this is indeed, possible.

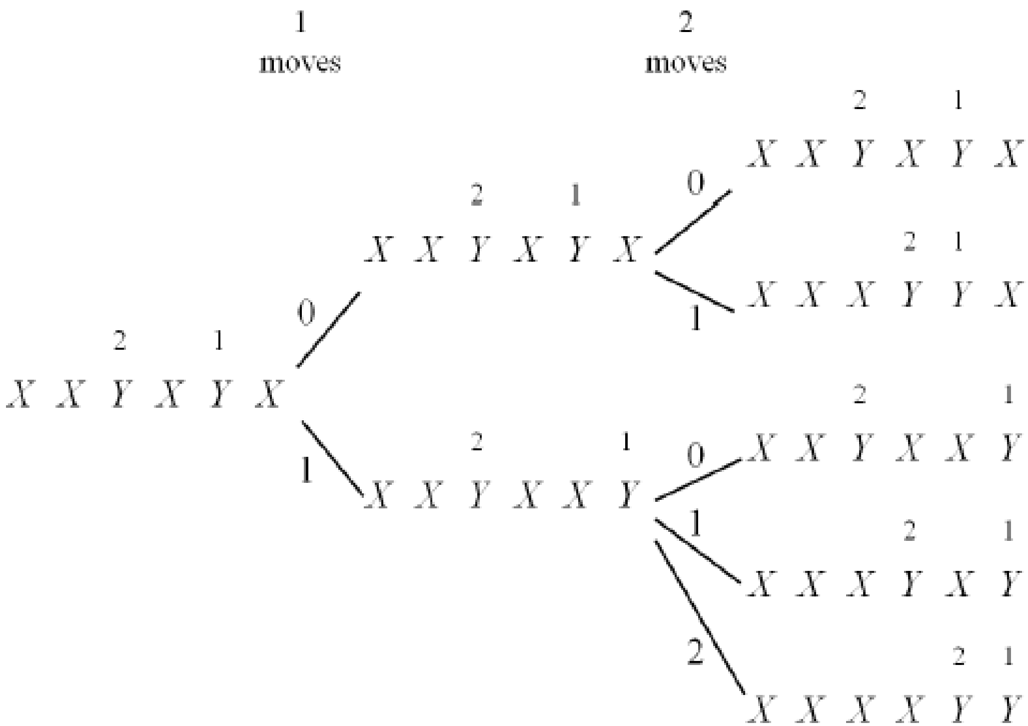
Lemma 1 *Let $\sigma_{N,W}$ be the sequence*

$$X(XY)^{\frac{N-W}{2}}X^{W-1}.$$

Then the good sequences of length N with a winning margin W for X are generated from $\sigma_{N,W}$ by all possible repeated exchanges of a Y with an X to its right.

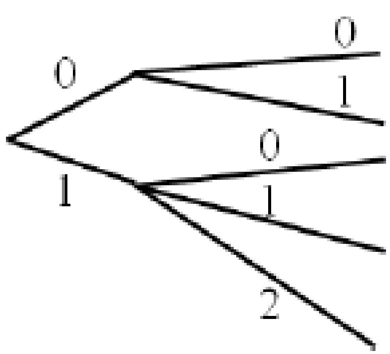
For example, $\sigma_{6,2}$ is sequence 1 in the table above. We may confirm that all the other sequences are the result of repeated right shifts of Y, starting from sequence 1.

Voting sequences 4

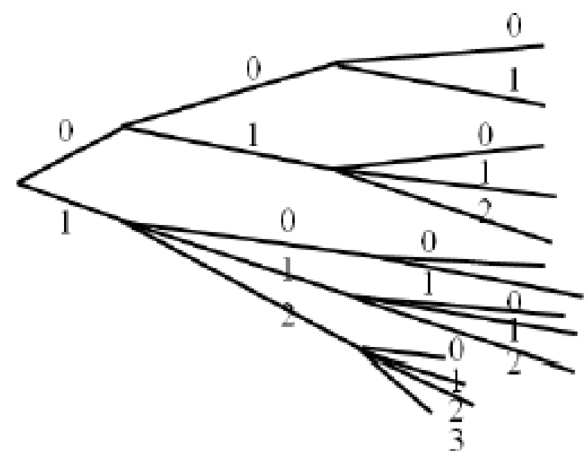


1	*X*	*X*	*Y*	*X*	*Y*	*X*
2	*X*	*X*	*X*	*Y*	*Y*	*X*
3	*X*	*X*	*Y*	*X*	*X*	*Y*
4	*X*	*X*	*X*	*Y*	*X*	*Y*
5	*X*	*X*	*X*	*X*	*Y*	*Y*.

Good sequences for N = 6, W = 2

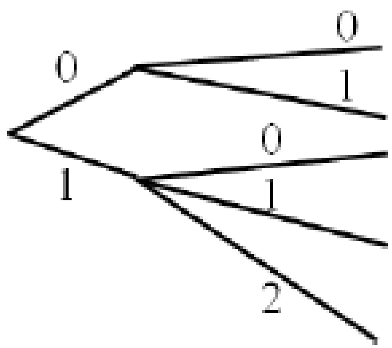


$\sigma_{6,2} = XXYXYX$;

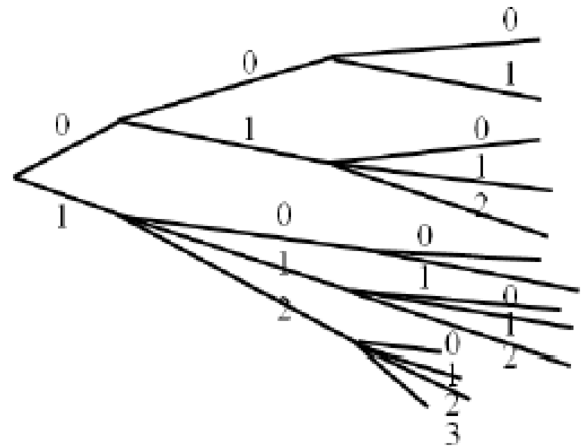


$\sigma_{8,2} = XXYXXYXYX$

Voting sequences 5



$\sigma_{6,2} = XXYXYX;$



$\sigma_{8,2} = XXYXYXYX$

These σ's all start with 2 *X*s just because *Y* must always be behind *X* in the vote.

If we ignore the leading *X* then we are talking about sequences where

The number of Ys never exceeds the number of Xs

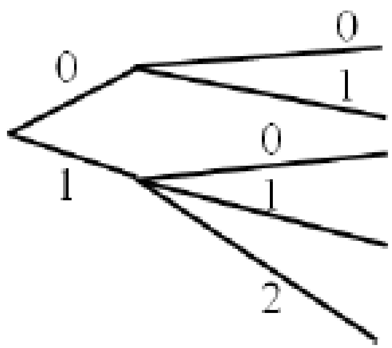
The trees here are winning margin of *W* = 2

Drop the initial *X*

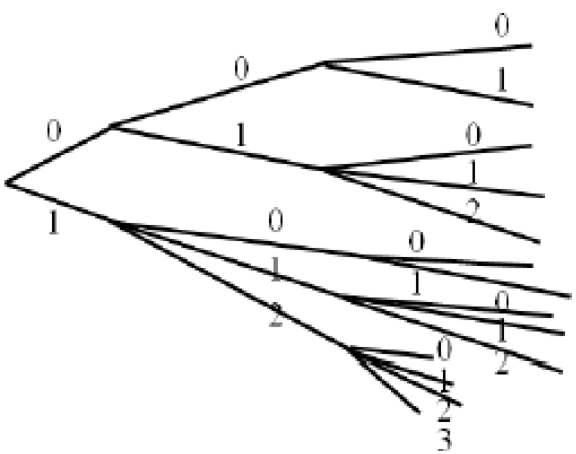
Add a *Y* at the end

*X*s now same as *Y*s. Same tree. Number of leaves is Catalan number!

Voting sequences 6



$\sigma_{6,2} = XXYXXYX;$



$\sigma_{8,2} = XXYXXYXXYX$

CATALAN ADDENDUM

Richard P. Stanley

version of 25 May 2013

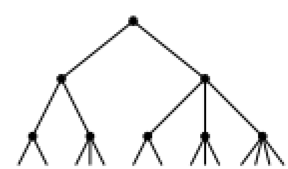
The problems below are a continuation of those appearing in Chapter 6 of *Enumerative Combinatorics*, volume 2. Combinatorial interpretations of Catalan numbers are numbered as a continuation of Exercise 6.19, while algebraic interpretations are numbered as a continuation of Exercise 6.25. Combinatorial interpretations of Motzkin and Schröder numbers are numbered as a continuations of Exercise 6.38 and 6.39, respectively. The remaining problems are numbered 6.C1, 6.C2, etc. I am grateful to Emeric Deutsch for providing parts (ooo), (a^4), (f^4), (z^4), (g^5), (p^5), (d^6), (u^7), (v^7), (x^7) and (y^7) of Exercise 6.19, and to Roland Bacher for providing (g^6).

a sequence of U's and D's), such that B is Dyck path and such that A and C have the same length

$$UUUDDD \quad UUUDDD \quad UUUDDD$$

$$UUDUDD \quad UDUDUD$$

(m^4) Vertices of height $n-1$ of the tree T defined by the property that the root has degree 2, and if the vertex x has degree k, then the children of x have degrees $2, 3, \dots, k+1$



(n^4) Motzkin paths (as de

More: math.mit.edu/~rstan/

Catalan's triangle 1

n \ k	0	1	2	3	4	5	6	7	8
0	1								
1	1	1							
2	1	2	2						
3	1	3	5	5					
4	1	4	9	14	14				
5	1	5	14	28	42	42			
6	1	6	20	48	90	132	132		
7	1	7	27	75	165	297	429	429	
8	1	8	35	110	275	572	1001	1430	1430

Wikipedia

C(4,2) = 9

XXXXYY
XXXYXY
XXXYYX
XXYXXY
XXYXYX
XXYYXX
XYXXXY
XYXXYX
XYXYXX

$C(n,k)$ is the number of sequences of n Xs and k Ys in which the number of Ys never exceeds the number of Xs.

'Completeable' bracket sequences, e.g. **( ) ( ) ((**

Observe $C(n,n) = C(n,n–1)$: every n,n bracket sequence is an $n,n–1$ sequence completed with a **)**

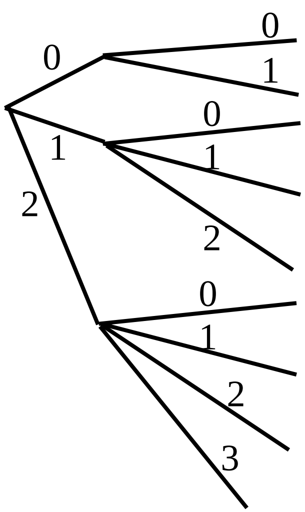
Catalan triangle 2

$C(4,2)$ = 4 Xs, 2 Ys

X Y X Y X X

Corresponds to ballot sequence of 5 Xs and 2 Ys, so $N = 7$, $W = 3$

X X Y X Y X X



n \ k	0	1	2	3	4
0	1				
1	1	1			
2	1	2	2		
3	1	3	5	5	
4	1	4	9	14	14
5	1	5	14	28	42
6	1	6	20	48	90

So:

Natural proof of the probability W/N for ballot sequences

same as counting good sequences

same as counting leaves in σ trees

same as calculating Catalan triangle entries

Catalan triangle formula 1

So I can still have my natural proof of the Ballot Theorem, by having a natural proof of

General formula [edit]

The general formula for $C(n,k)$ is given by[1][4]

$$C(n,k) = \binom{n+k}{k} - \binom{n+k}{k-1}$$

So

$$C(n,k) = \frac{n-k+1}{n+1}\binom{n+k}{k}$$

When $k = n$, the diagonal $C(n,n)$ is the n-th Catalan number.

n \ k	0	1	2	3	4	5	6
0	1						
1	1	1					
2	1	2	2				
3	1	3	5	5			
4	1	4	9	14	14		
5	1	5	14	28	42	42	
6	1	6	20	48	90	132	132

Wikipedia

Counting Arrangements of 1's and -1's

D. F. Bailey

Mathematics Magazine, Vol. 69, No. 2 (Apr., 1996), pp. 128-131

Catalan triangle formula 2

Counting Arrangements of 1's and −1's

D. F. BAILEY
Trinity University
San Antonio, TX 78212

It is well known that the nth Catalan number counts the number of sequences with non-negative partial sums that can be formed from n 1's and n −1's. (See [1].) In this paper we derive a formula for the number of such sequences formed from n 1's and k −1's. In the process we produce a non-standard proof that the nth Catalan number is given by

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

Because the numbers we define can be evaluated in a manner similar to the binomial coefficients, we use a symbolism reminiscent of the standard notation for the binomial coefficients.

Definition. Let $\left\{ n \atop k \right\}$ denote the number of arrangements of n 1's and k −1's $a_1 a_2 \dots a_{n+k}$ so that

$$a_1 + a_2 + \cdots + a_i \geq 0 \text{ for all } 1 \leq i \leq n+k.$$

Counting Arrangements of 1's and -1's

D. F. Bailey

Mathematics Magazine, Vol. 69, No. 2 (Apr., 1996), pp. 128-131

LEMMA 1.

(i) $\left\{ n \atop 0 \right\} = 1$ *for* $n \geq 0$.

(ii) $\left\{ n \atop 1 \right\} = n$ *for* $n \geq 1$.

(iii) $\left\{ n+1 \atop k \right\} = \left\{ n+1 \atop k-1 \right\} + \left\{ n \atop k \right\}$ *for* $1 < k < n+1$.

(iv) $\left\{ n+1 \atop n+1 \right\} = \left\{ n+1 \atop n \right\}$ *for* $n \geq 1$.

LEMMA 2. *For* $2 \leq k \leq n$ *we have*

$$\left\{ n \atop k \right\} = \sum_{i=k}^{n} \left\{ i \atop k-1 \right\}.$$

LEMMA 3. *If* $n \geq 2$ *then*

$$\left\{ n \atop 2 \right\} = \frac{(n-1)(n+2)}{2}.$$

LEMMA 4.

$$\sum_{i=k+1}^{n} (i+1-k)(i+2)(i+3)\dots(i+k) = \frac{1}{k+1}(n-k)(n+2)(n+3)\dots(n+1+k).$$

THEOREM. *For* $n \geq k \geq 2$

$$\left\{ n \atop k \right\} = \frac{(n+1-k)(n+2)(n+3)\dots(n+k)}{k!}.$$

$$= \frac{n+1-k}{n+1}\binom{n+k}{k}$$

Catalan numbers again 1

My aim now: a natural proof of:

$$C(n,k) = \binom{n+k}{k} - \binom{n+k}{k-1}$$

Surely, this is

$$\binom{n+k}{k} - \binom{n+k}{k-1}$$

All ways of placing *k* *Y*s into *n* + *k* positions

All bad ways of placing *k* *Y*s into *n* + *k* positions

Catalan numbers again 2

$$\binom{n+k}{k} - \binom{n+k}{k-1}$$

All ways of placing k Ys into $n + k$ positions

All bad ways of placing k Ys into $n + k$ positions

Surely, the bad sequences are:

$$\binom{n+k}{k-1} = \binom{n+k-1}{k-1} + \binom{n+k-1}{k-2}$$

Start with a Y. The remaining k–1 Ys can go anywhere in the remaining n+k–1 places.

Start with an X. There are k–2 choices to make in the remaining n+k–1 places. What choices?

Catalan numbers again 3

Surely, the bad sequences are:

$$\binom{n+k}{k-1} = \binom{n+k-1}{k-1} + \binom{n+k-1}{k-2}$$

???

In fact, I think I fell into the trap of thinking a binomial coefficient must correspond to a bijection.

I've come to the conclusion that the natural identity for bad sequences is:

$$\binom{n+k}{k-1} = \sum_{i=0}^{k-1} C_i \binom{n+k-1-2i}{k-1-i}$$

↖ note, first term is $\binom{n+k-1}{k-1}$

Catalan numbers again 4

$$\binom{n+k}{k-1} = \sum_{i=0}^{k-1} C_i \binom{n+k-1-2i}{k-1-i}$$

All bad ways of placing k Ys into $n + k$ positions

Bad after 1st entry. Happens 1 way (starts Y)
+ bad after 2nd entry. In 1 way
+ bad after 3rd entry. In C_2 ways
+ bad after 4th entry. In C_3 ways
+ bad after 5th entry. In C_4 ways

...

Corresponds to an infinite tree that constructs all the bad sequences.
The number of leaves at height n is C_{n-1} another construction of Catalan numbers?!