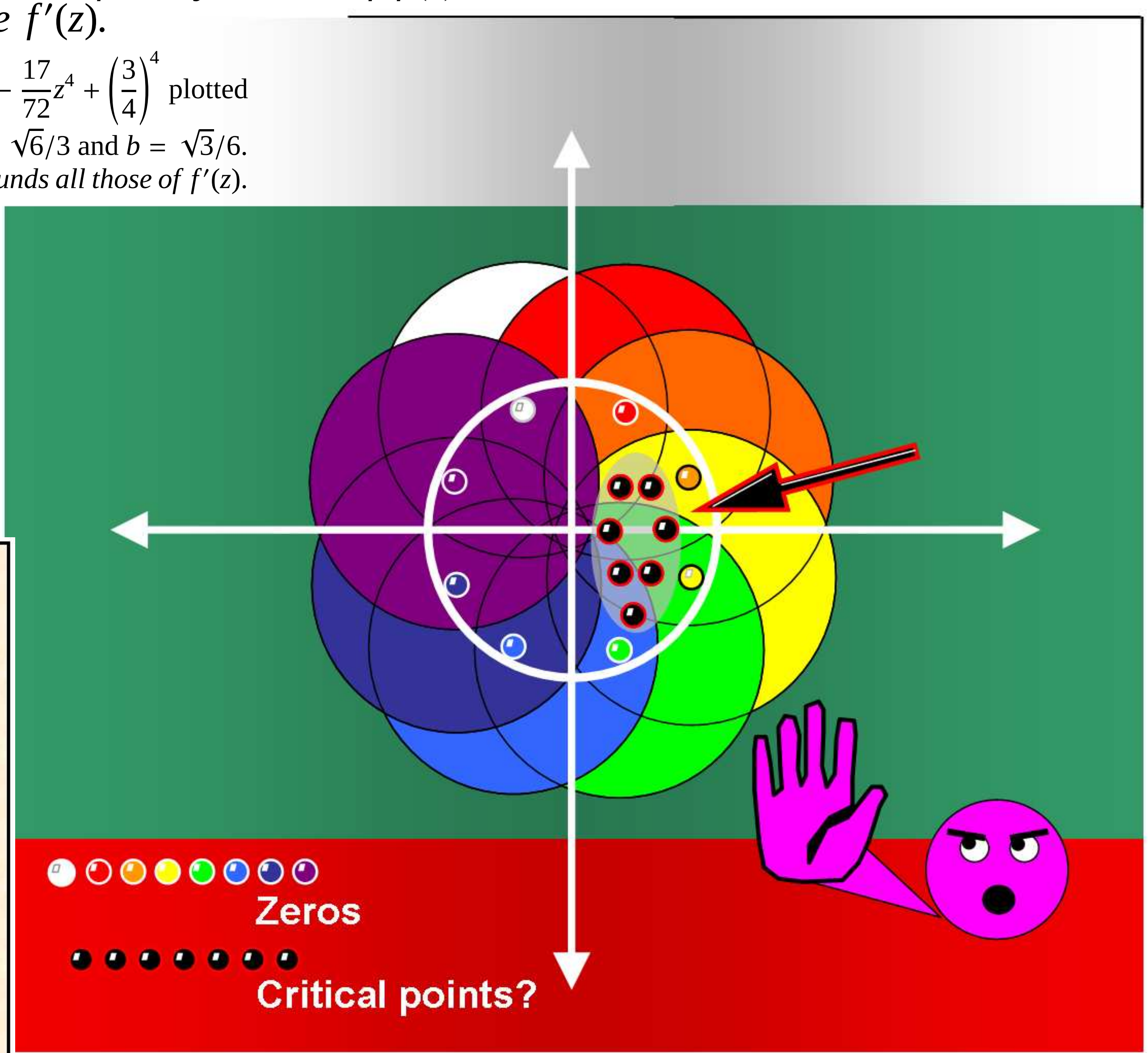
THEOREM OF THE DAY

true for low degree ($n \le 8$) & high degree

Sendov's Conjecture (a Theorem Under Construction!) *Let $f(z)$ be a polynomial of degree $n \ge 2$, all of whose zeros lie in the closed unit disk. Then for any zero z_0 of $f(z)$, the closed unit disk with centre z_0 contains at least one zero of the derivative $f'(z)$.*



The sketch on the right shows the zeros of the polynomial $f(z) = z^8 - \frac{17}{72}z^4 + \left(\frac{3}{4}\right)^4$ plotted on the complex plane: these zeros are $\pm a \pm bi$ and $\pm b \pm ai$, where $a = \sqrt{6}/3$ and $b = \sqrt{3}/6$. By the **Gauss–Lucas Theorem**: *the convex hull of the zeros of $f(z)$ bounds all those of $f'(z)$*. So $f'(z)$ certainly has no zeros outside the unit circle depicted centered at the origin. Perhaps these so-called 'critical points' could, however, cluster on one side of the unit circle, as suggested in the sketch? No: all of our suggested critical points lie outside the unit circles centered at the two leftmost zeros of $f(z)$; Sendov's Conjecture, known to be true for polynomials of degree 8, and for high degree, asserts this cannot happen. In fact, $f'(z)$ has three zeros at the origin, lying within unit distance of all the zeros of $f(z)$ (the other four zeros are the fourth roots of 17/144, being approx. ± 0.59 and $\pm 0.59i$).

Construction notes:

1959: The Bulgarian mathematician Blagovest Sendov first proposes his conjecture to Nikola Obreschkov.

1967: Due to a misapprehension, the conjecture is publicised as belonging to fellow Bulgarian Ljubomir Iliev by the influential analyst Walter Kurt Hayman; it becomes widely known as Iliev's (or Ilieff's) Conjecture.

1969: Amram Meir and Ambikeshwar Sharmaand prove the conjecture for polynomials of degree $n \le 5$.

1991: After 20 years, Johnny E. Brown of Purdue University pushes on to $n \le 6$.

1996: Julius Borcea achieves $n \le 7$; Johnny E. Brown's independent proof appears a year later.

1999: Brown and his PhD student Guangping Xiang reach $n \le 8$, now a record of nearly 20 years standing!

2011: Jérôme Dégot proves the conjecture for large degree (depending on z_0).

Web link: www.math.bas.bg/serdica/n4_02.html: the paper by Sendov.

Further reading: *Polynomials* by Victor Prasolov, Springer, 2nd printing, 2009.