THEOREM OF THE DAY

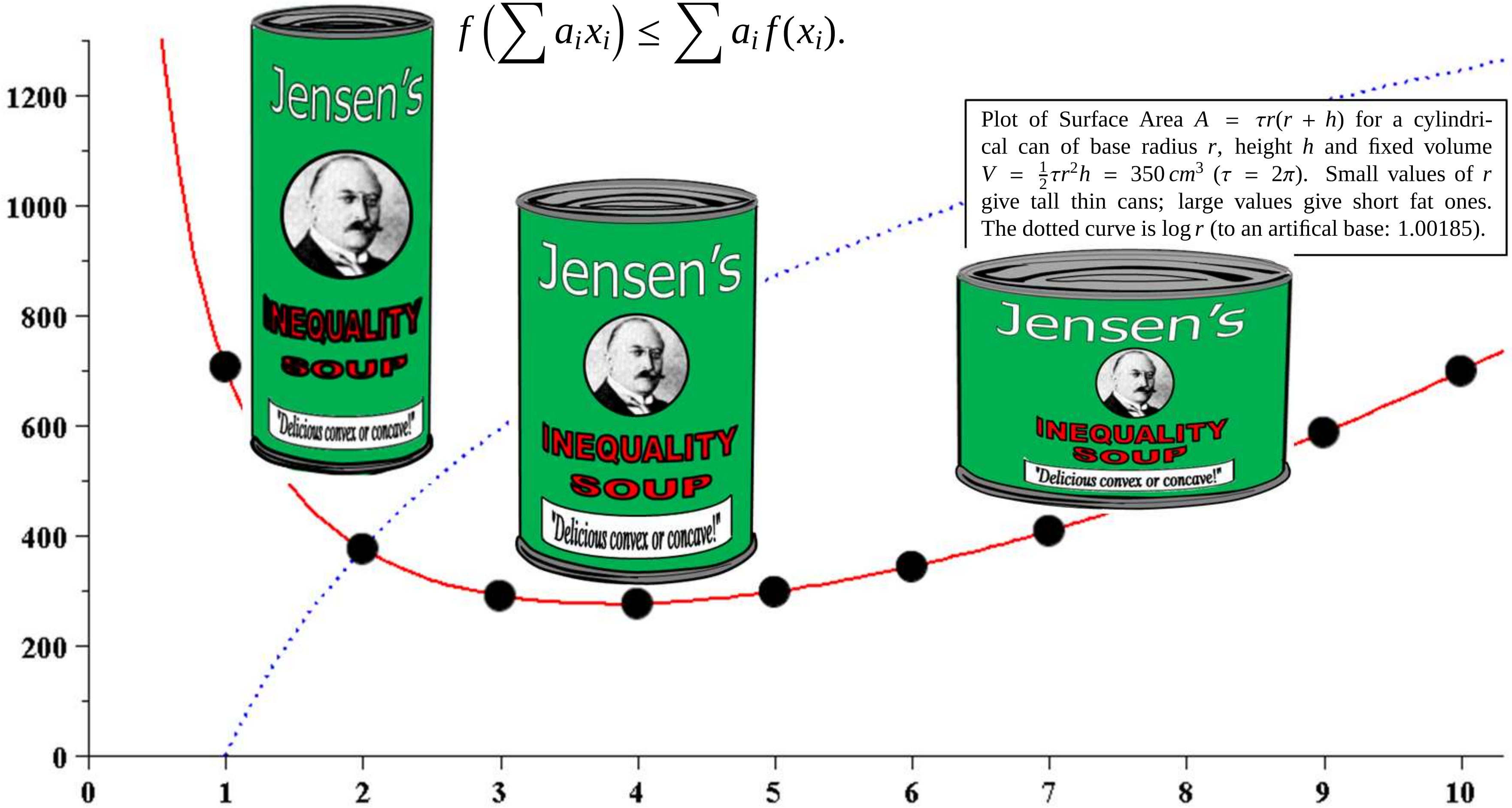
Jensen's Inequality *If f is a real convex function, $x_1, \ldots, x_n$ are values in the domain of f, and $a_1, \ldots, a_n$ are positive real numbers summing to 1 then*

$$f\left(\sum a_i x_i\right) \leq \sum a_i f(x_i).$$

The function plotted here, showing relationship of radius in *cm* (horizontal axis) to surface area in cm^2 (vertical axis) is a **convex function**: for any two points on the plotted curve, all internal points on the straight line joining them lie above the curve. If 'above' is replaced by 'below' then the function is **concave**, e.g. the $\log r$ curve, and Jensen's Inequality is reversed, with the $\sum a_i f(x_i)$ on the left.

Jensen's Inequality is very general and has many applications, a classic example being the proof of the **Arithmetic-Geometric Mean (AM-GM) Inequality:**

$$\sqrt[n]{\Pi x_i} \leq \frac{1}{n}\sum x_i. \qquad (1)$$



Plot of Surface Area $A = \tau r(r+h)$ for a cylindrical can of base radius r, height h and fixed volume $V = \frac{1}{2}\tau r^2 h = 350\,cm^3$ ($\tau = 2\pi$). Small values of r give tall thin cans; large values give short fat ones. The dotted curve is $\log r$ (to an artifical base: 1.00185).

Indeed, using the concave function $\log x$, with all the a_i set equal to $1/n$: $\sum\left(\frac{1}{n}\log(x_i)\right) \leq \log\sum\left(\frac{1}{n}x_i\right)$. Then $\frac{1}{n}\log(\Pi x_i) = \log\left(\sqrt[n]{\Pi x_i}\right) \leq \log\left(\frac{1}{n}\sum x_i\right)$, and exponentiation on both sides gives (1).

For **strictly** convex or concave functions, i.e. with the gradient never changing sign, equality in Jensen's Inequality holds if and only if all the x_i are equal. Since $\log x$ is strictly concave this is true for the AM-GM Inequality, allowing a pretty, non-calculus derivation of minimum surface area A for our soup can. Thus: $A = \tau r^2 + \tau r h = \tau r^2 + 2V/r = \frac{1}{3}(3\tau r^2 + 3V/r + 3V/r)$, viewed as an arithmetic mean of three quantities. Apply the AM-GM Inequality: $\sqrt[3]{3\tau r^2 \times 3V/r \times 3V/r} = 3\sqrt[3]{\tau V^2} \leq A$, with equality if and only if $\tau r^2 = V/r = \frac{1}{2}\tau r h$, giving $h = 2r$. For our $350\,cm^3$ can, the optimal radius is about $3.82\,cm$ giving $A \approx 275\,cm^2$.

This inequality is embodied in Hölder's Defect Formula of 1885; Johan Jensen provided its general form in terms of convex functions in 1906.

Web link: www.math.ualberta.ca/pi/pastissues.html: Dec. 2001 & 2002. And jwilson.coe.uga.edu/: "Using the Arithmetic Mean-Geometric Mean Inequality...".

Further reading: *The Cauchy-Schwarz Master Class: An Introduction to the Art of Mathematical Inequalities* by J. Michael Steele, Mathematical Association of America, 2004, chapter 6.