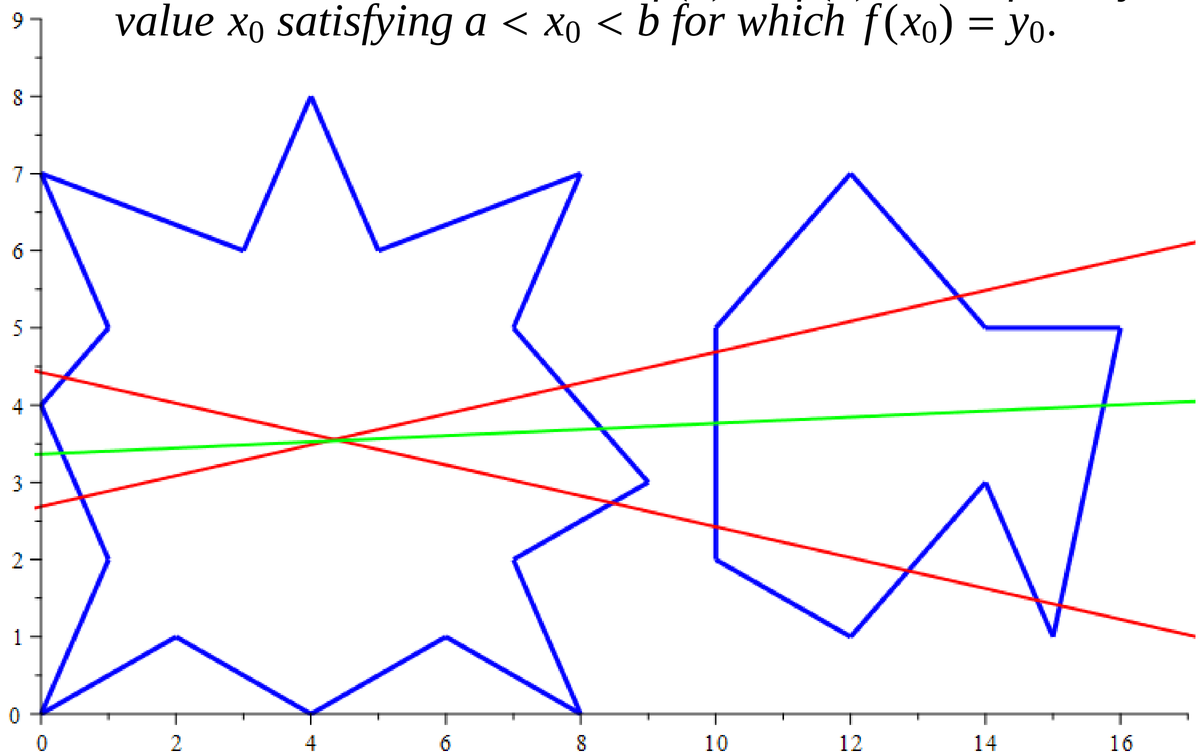
THEOREM OF THE DAY

The Intermediate Value Theorem *Let $f(x)$ be a real-valued function which is continuous on the closed interval $[a, b]$ and such that $f(a) < f(b)$. Then for any value y_0 satisfying $f(a) < y_0 < f(b)$, there is a value x_0 satisfying $a < x_0 < b$ for which $f(x_0) = y_0$.*



In our illustration above, the three straight lines bisect the left-hand pancake. One crosses the right-hand pancake too low down to bisect its area and another crosses too high. But the middle straight line bisects and achieves the conclusion of the Two Pancakes Theorem. To be precise, however, this right-hand bisection is approximate. Indeed, the Intermediate Value Theorem itself is non-constructive, and in any particular application it may or may not be the case that a direct construction of an intermediate value is possible. Thus, there are known exact methods for bisecting certain types of polygon in a given direction and these are being applied above to the left-hand pancake. But exact bisection of two pancakes is not in general available: the intermediate value exists but must apparently be approximated.

The Pancake Theorem *If P is a simple closed curve in the plane, then for any specified angle there is a unique straight line at this angle to the horizontal which bisects the area of P.*
Proof. Let P have area A. Let the specified angle be given as θ to the horizontal. Let a straight line at angle θ lie entirely below P. Define $f(y)$ to be the function which records the area of that part of curve P which lies below the line when it is translated vertically by y. Then $f(0) = 0$, $f(h) = A$, for some sufficiently high value of h, and f is continuous on the interval $[0, h]$. So the Intermediate Value Theorem says there is height h_0 with $f(h_0) = A/2$. Moreover, f is a strictly increasing function and therefore there is a unique line at angle θ bisecting P_1.

The Two Pancakes Theorem *Let P_1 and P_2 be simple closed curves in the plane. Then there is a straight line in the plane which simultaneously bisects the area of both P_1 and P_2.*
Proof. We prove a version in which P_2 is entirely to the right of P_1 in the positive quadrant. Let the area of P_2 be A. Define $g(\theta)$ to be the function which records the area of that part of curve P_2 which lies below the unique straight line at angle θ to the horizontal which bisects the area of P_1. Then it is clear that $g(\theta_1) = 0$ and $g(\theta_2) = A$ for suitably chosen angles $-\tau/4 \leq \theta_1$ and $\theta_2 \leq \tau/4$. Then, subject to a proof that g is continuous on the interval $[\theta_1, \theta_2]$, the Intermediate Value Theorem confirms that there is a value θ_0 for which $g(\theta_0) = A/2$.

We may give credit to Bernhard Bolzano for the first exploration (1817) of the intermediate value property as a consequence of a rigorously defined notion of continuity. Only by the mid-nineteenth-century did our theorem occupy a secure place in modern analysis, before being queried anew by the intuitionists in the early twentieth century.

Web link: prove-me-wrong.com/2022/07/01/the-pancake-theorem/

Further reading: *A Radical Approach to Real Analysis, 2nd edition* by David M. Bressoud, Mathematical Association of America, 2007, chapter 3, section 3.