Polygons and a hash function

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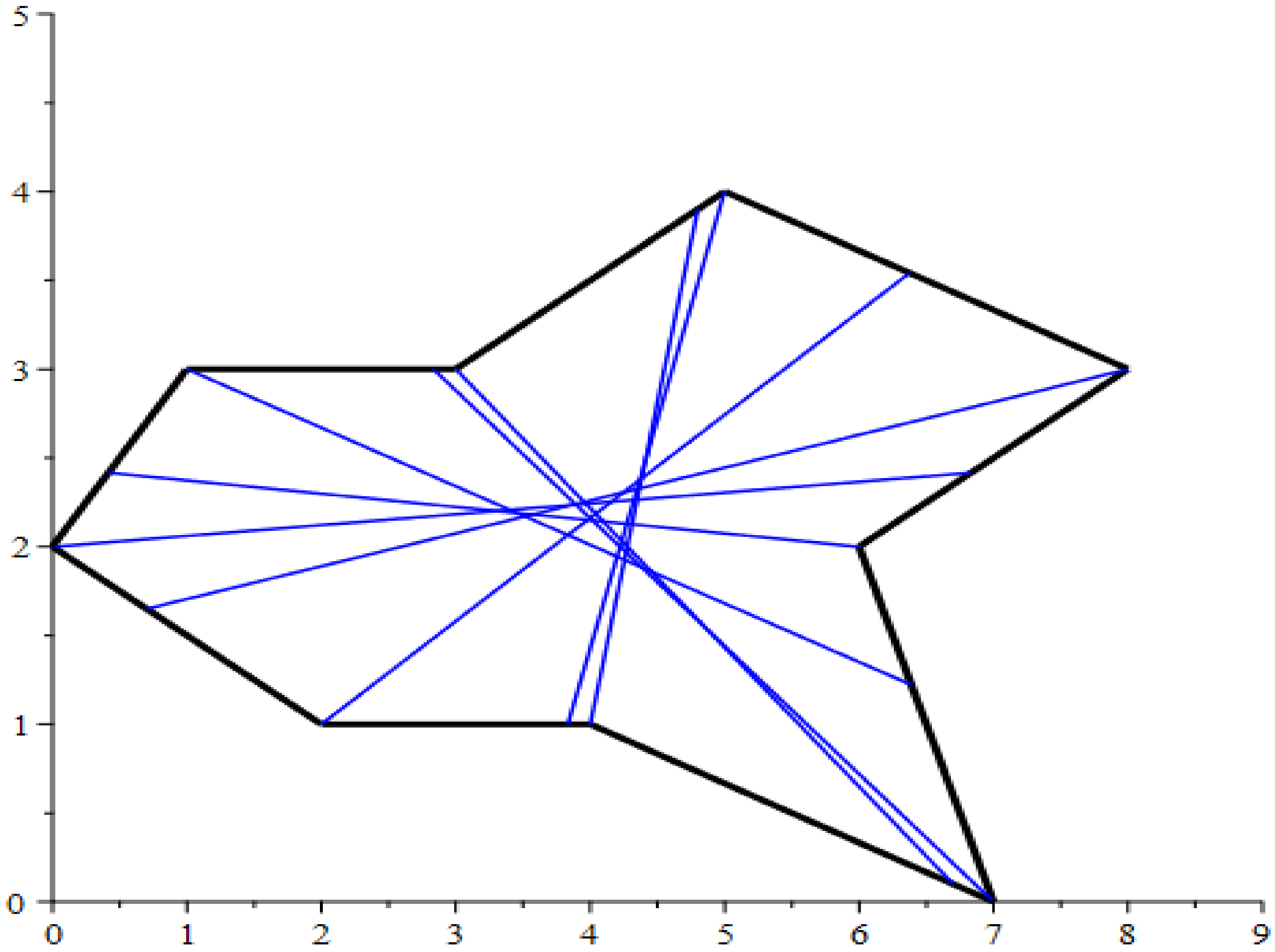
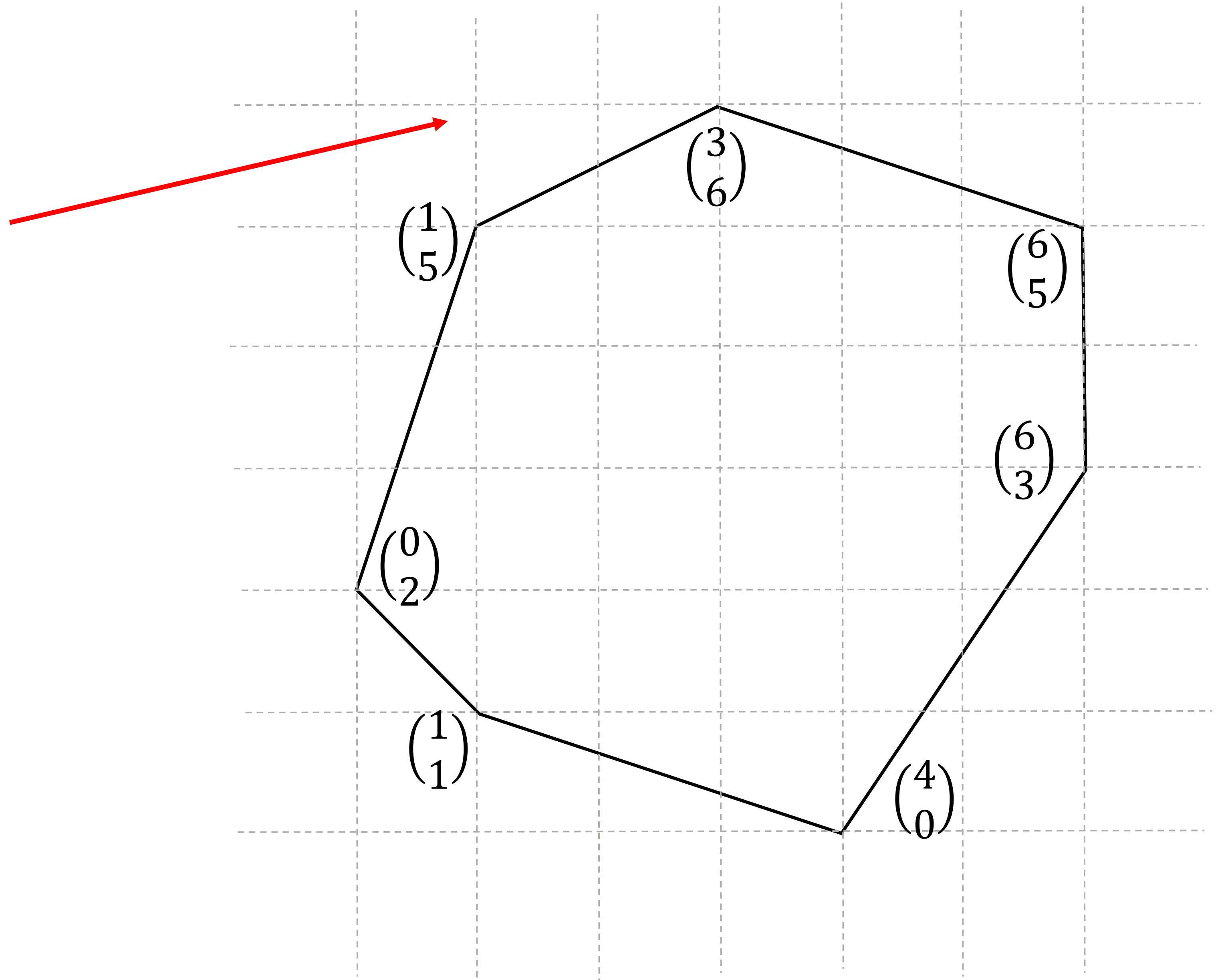
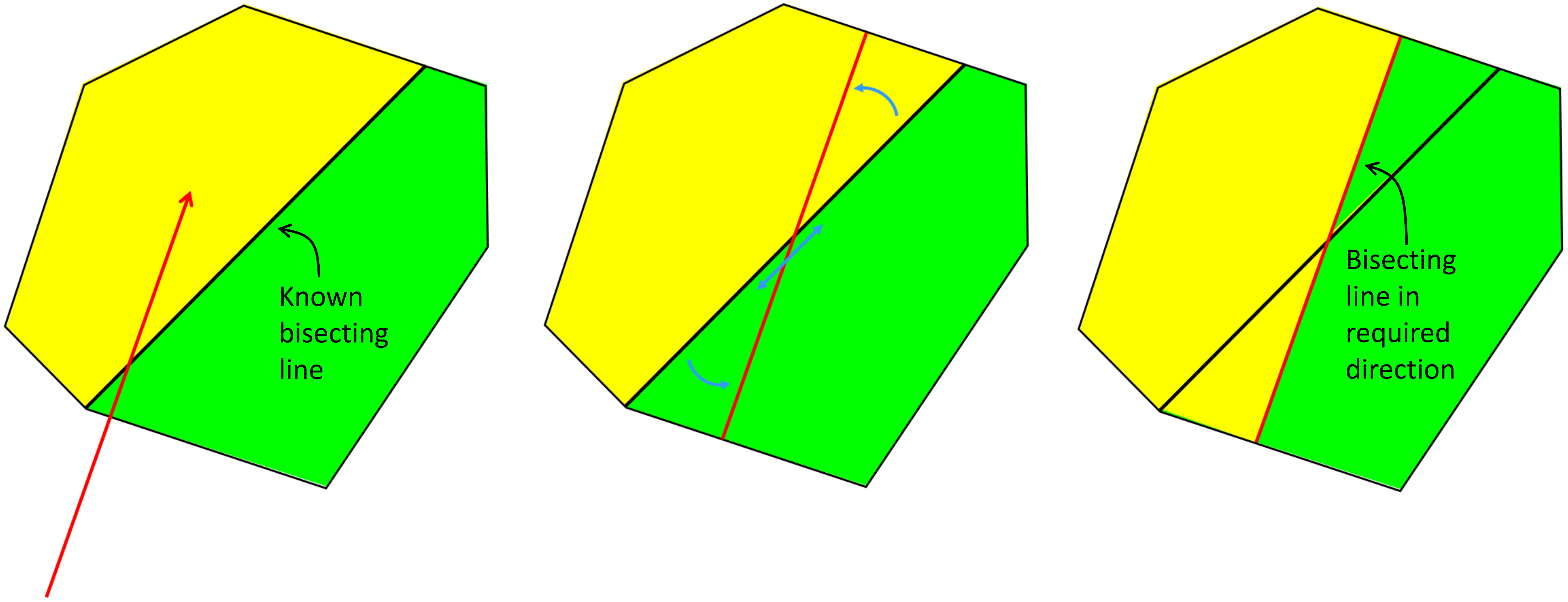


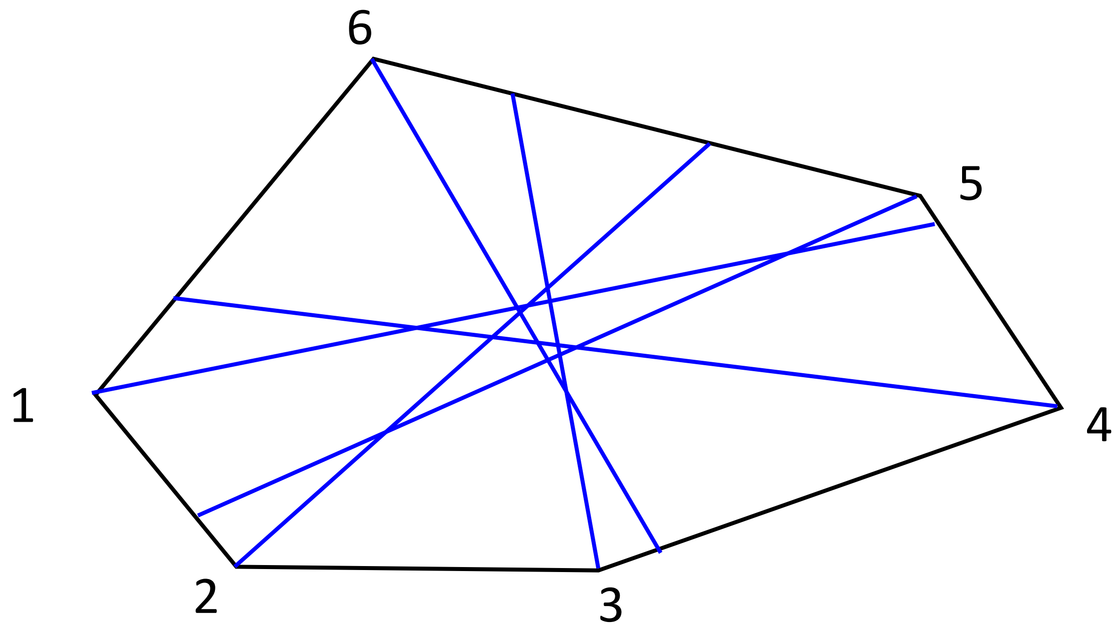
Bisecting convex polygons



A straight line equation for convex polygon bisection…

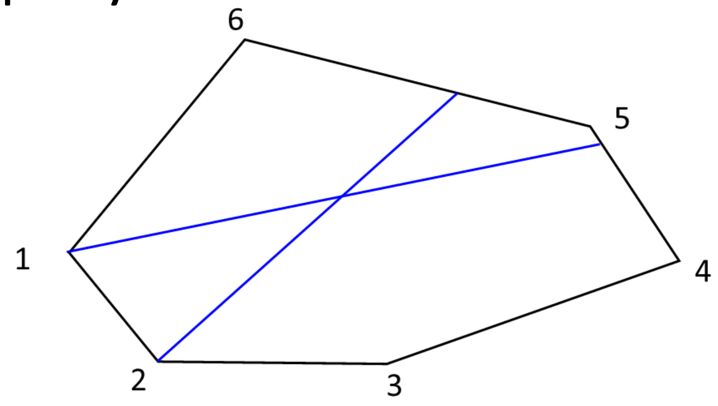


Pairs of bisecting lines



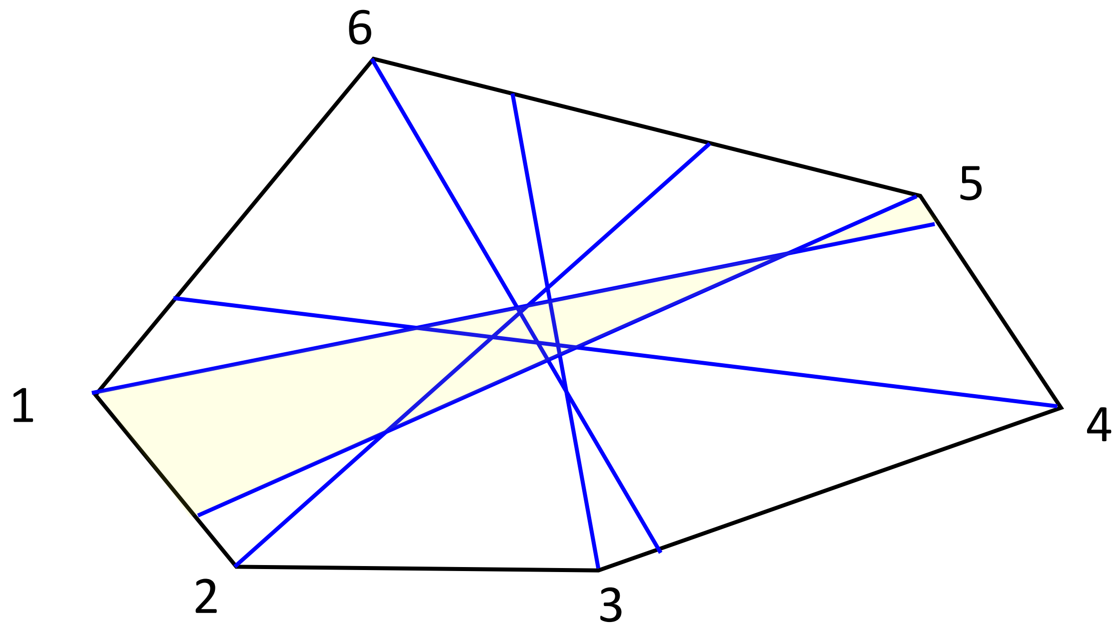
1, 4
2, 5
3, 5
4, 6
5, 1
6, 3

We want to take consecutive pairs of bisecting lines, forming a sector of the circle.
We want to avoid sectors which properly contain a vertex.



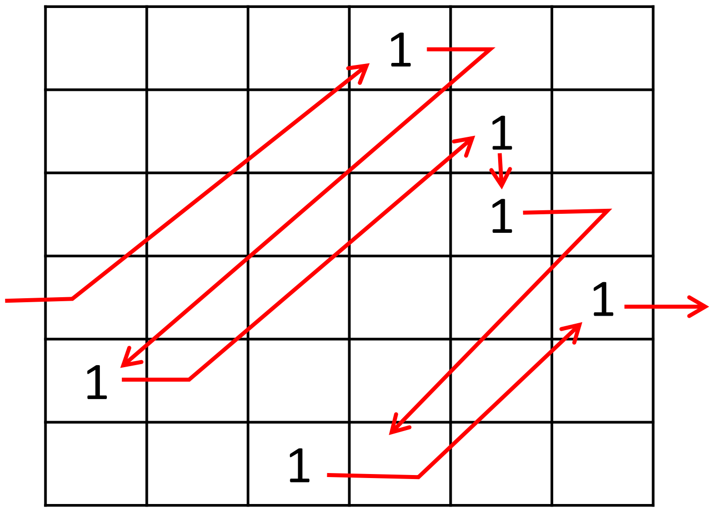
So we want to take the lines in an ordering which avoids this

Ordering bisecting line pairs



1, 4		1, 4
2, 5		5, 1
3, 5	→	2, 5
4, 6		3, 5
5, 1		6, 3
6, 3		4, 6

Represent the lines as matrix entries. The matrix lives on a torus, so it wraps round horizontally and vertically.



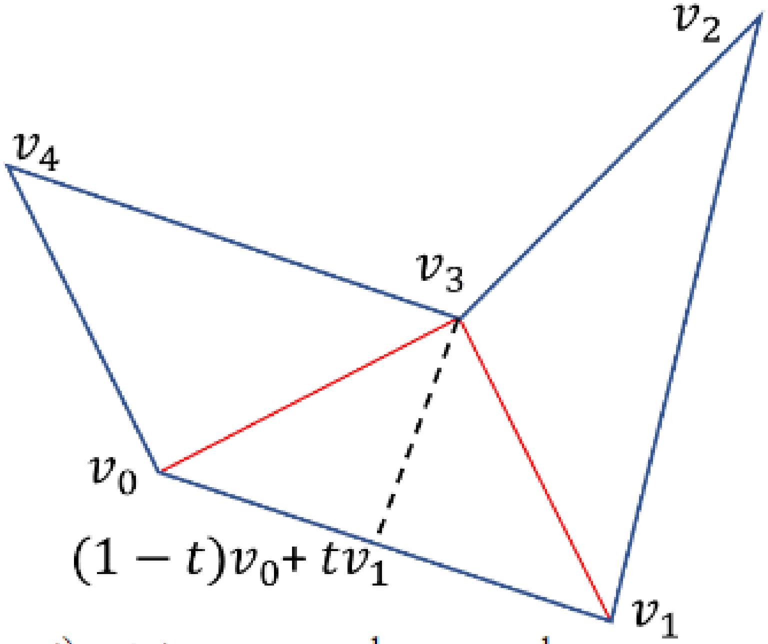
Connect the entries in a cycle using the following rules

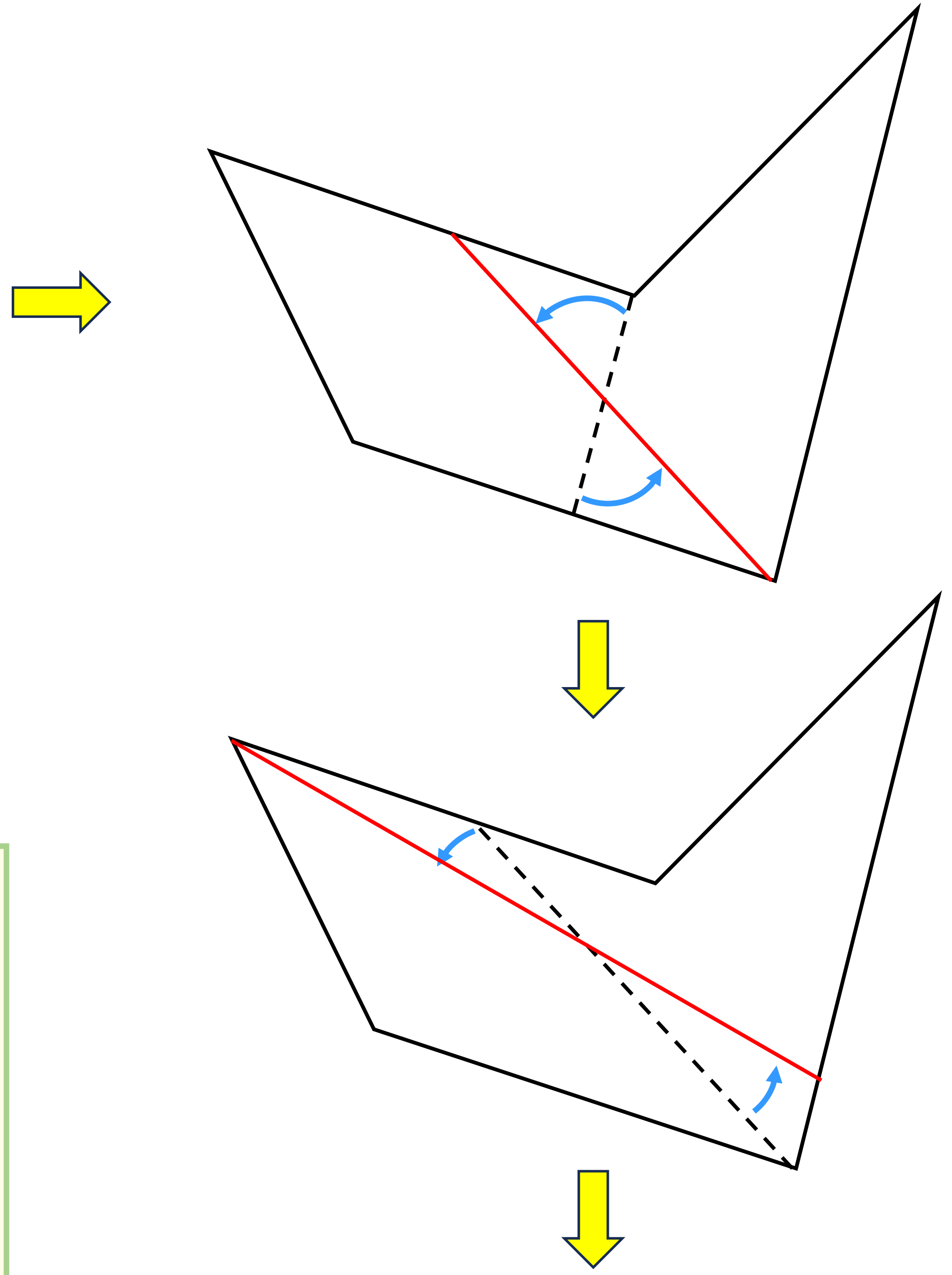
1. a column of 1s is joined vertically
2. a 1 with a vacant cell below is joined to the entry diagonally opposite the cell to its right.

The result is the ordering we want:

Finding the bisecting lines

An Application
We may triangulate a polygon on n vertices by adding $n-3$ diagonals, as illustrated on the right. We would like to test if some straight line joining a triangle vertex to the opposite polygon edge bisects the area of the polygon. In our diagram this requires a value of $t \in [0,1]$ for which the polygons $v_0, (1-t)v_0 + tv_1, v_3, v_4$ and $(1-t)v_0 + tv_1, v_1, v_2, v_3$ have equal area.



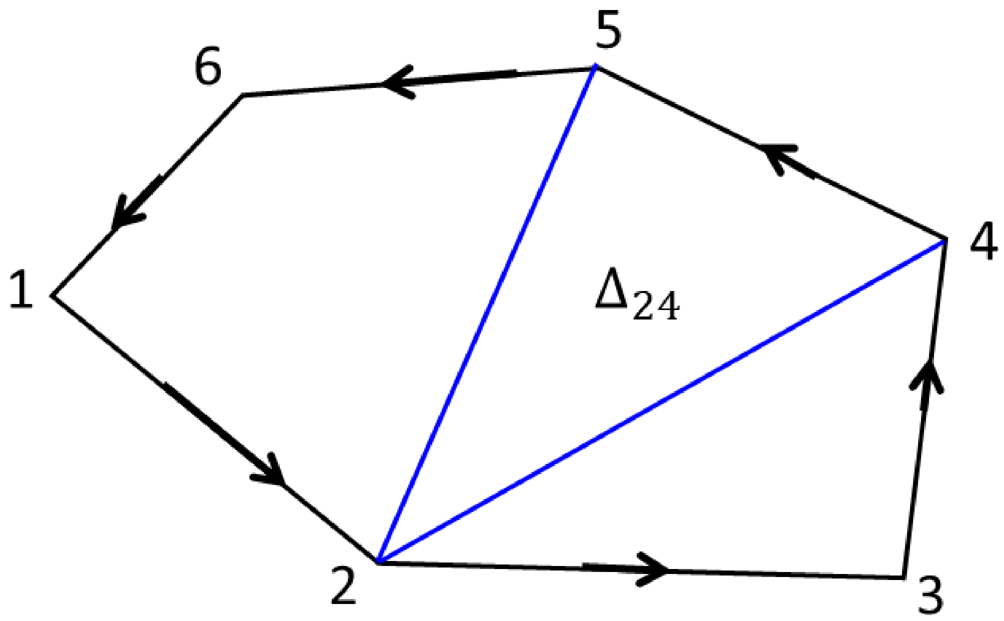


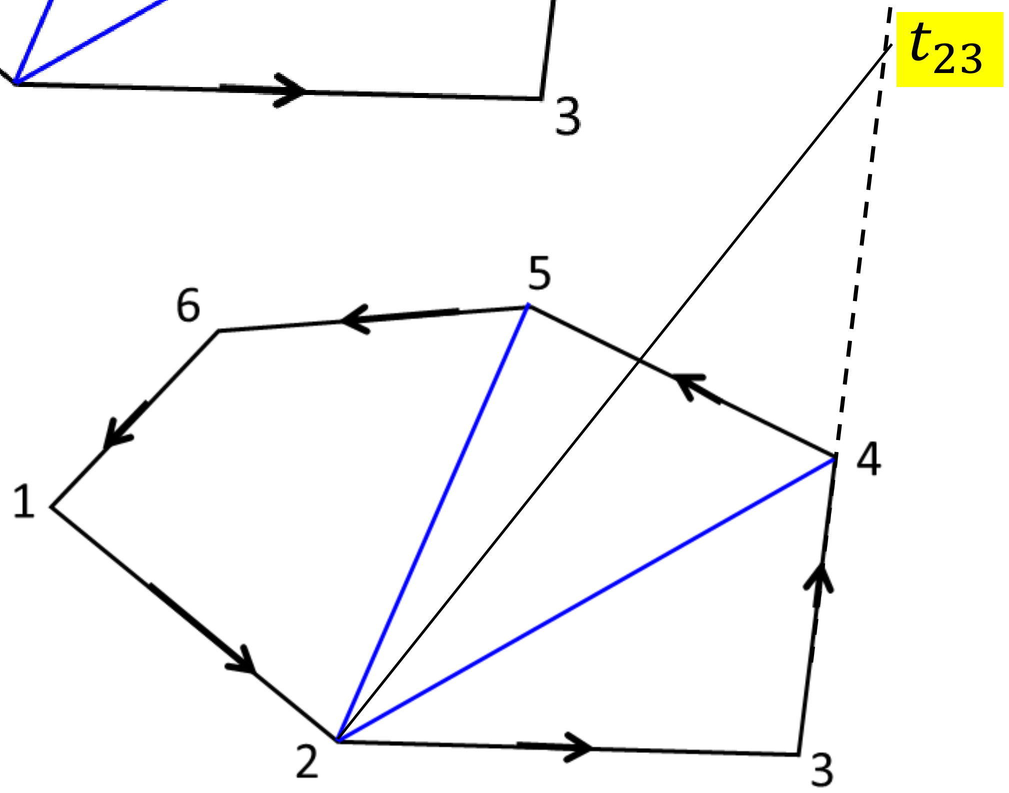
An application of the shoelace formula gives:

$$t = \frac{A_{co} - A_{cl}}{2A_{\Delta}}$$

A_{co} =area counter-clockwise from and including middle triangle
A_{cl} =area clockwise from middle triangle
A_{Δ} =area of middle triangle

Conventions/notation





Polygons are oriented anticlockwise.

Vertices labelled $1, \ldots, n$.

Triangle from vertex i to opposite edge $p, p+1$ is denoted Δ_{ip}

There are $n-2$ triangles opposite, say, vertex 3 which are

$$\Delta_{34}, \Delta_{35}, \ldots, \Delta_{3,3+n-2}$$

taken mod n. (The mod arithmetic, since I'm insisting on counting from 1, is

$$i \to 1 + (i - 1 \bmod n).$$

The bisection point on the (extended) edge $p, p+1$ opposite vertex i is denoted t_{ip}.

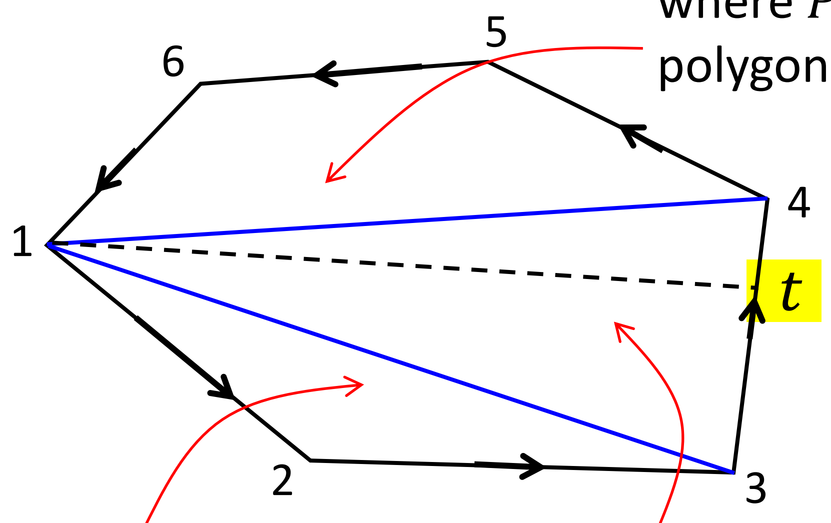
Proceeding systematically

An application of the shoelace formula gives:

$$t = \frac{A_{co} - A_{cl}}{2A_\Delta}$$

A_{co} =area counter-clockwise from and including middle triangle
A_{cl} =area clockwise from middle triangle
A_Δ =area of middle triangle

$A_{co} = P - \Delta_{12}$ where P is polygon area



$A_{cl} = \Delta_{12}$ $A_\Delta = \Delta_{13}$

$$t = \frac{P - \Delta_{12} - \Delta_{12}}{2\Delta_{13}}$$

In what follows we take triangles round our polygon in a rather counter-intuitive ordering so as to simplify notation.

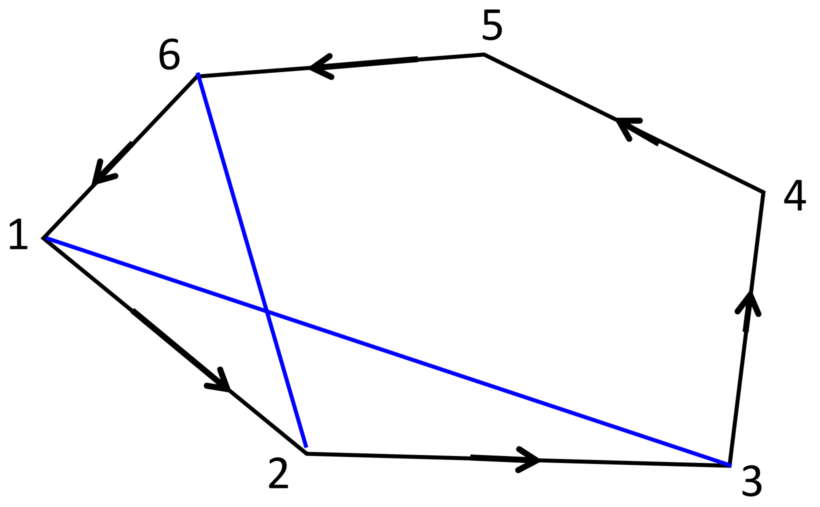
$$t_{15} = \frac{\Delta_{15} - (P - \Delta_{15})}{2\Delta_{15}} = 1 - \frac{P}{2\Delta_{15}}$$

$$t_{14} = \frac{\Delta_{15} + \Delta_{14} - (P - \Delta_{15} - \Delta_{14})}{2\Delta_{14}} = \frac{\Delta_{15} + \Delta_{14} + (2t_{15}\Delta_{15} - \Delta_{15}) + \Delta_{14}}{2\Delta_{14}} = 1 + t_{15}\frac{\Delta_{15}}{\Delta_{14}}$$

$$t_{13} = \frac{\Delta_{15} + \Delta_{14} + \Delta_{13} - (P - \Delta_{15} - \Delta_{14} - \Delta_{13})}{2\Delta_{13}} = \frac{\Delta_{15} + \Delta_{14} + \Delta_{13} + (2t_{14}\Delta_{14} - \Delta_{15} - \Delta_{14}) + \Delta_{13}}{2\Delta_{13}} = 1 + t_{14}\frac{\Delta_{14}}{\Delta_{13}}$$

$$\cdots$$

Zigzagging



OR: $$t_{12} = \frac{P - 0}{2\Delta_{12}}$$

$$t_{26} = 1 - \frac{P}{2\Delta_{26}} = 1 - \frac{2t_{12}\Delta_{12}}{2\Delta_{26}}$$

So: $$t_{15} = 1 - \frac{P}{2\Delta_{15}}$$

$$t_{14} = 1 + t_{15}\frac{\Delta_{15}}{\Delta_{14}}$$

$$t_{13} = 1 + t_{14}\frac{\Delta_{14}}{\Delta_{13}}$$

And similarly: $$t_{12} = 1 + t_{13}\frac{\Delta_{13}}{\Delta_{12}}$$

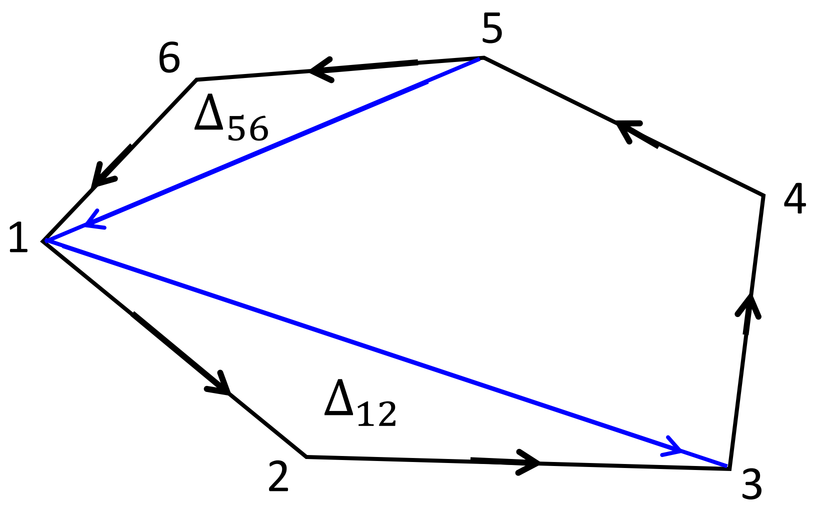
$$t_{26} = 1 - t_{12}\frac{\Delta_{12}}{\Delta_{26}}$$

And similarly: $$t_{25} = 1 + t_{26}\frac{\Delta_{26}}{\Delta_{25}}$$

$$t_{24} = 1 + t_{25}\frac{\Delta_{25}}{\Delta_{24}}$$

…

All t values from a matrix of triangle areas



	1	2	3	4	5	6
1		Δ_{12}	Δ_{13}	Δ_{14}	Δ_{15}	
2			Δ_{23}	Δ_{24}	Δ_{25}	Δ_{26}
3	Δ_{31}			Δ_{34}	Δ_{35}	Δ_{36}
4	Δ_{41}	Δ_{42}			Δ_{45}	Δ_{46}
5	Δ_{51}	Δ_{52}	Δ_{53}			Δ_{56}
6	Δ_{61}	Δ_{62}	Δ_{63}	Δ_{64}		

$$t_{15} = 1 - \frac{P}{2\Delta_{15}}$$

$$t_{14} = 1 + t_{15}\frac{\Delta_{15}}{\Delta_{14}}$$

$$t_{13} = 1 + t_{14}\frac{\Delta_{14}}{\Delta_{13}}$$

$$t_{12} = 1 + t_{13}\frac{\Delta_{13}}{\Delta_{12}}$$

$$t_{26} = 1 - t_{12}\frac{\Delta_{12}}{\Delta_{26}}$$

$$t_{25} = 1 + t_{26}\frac{\Delta_{26}}{\Delta_{25}}$$

$$t_{24} = 1 + t_{25}\frac{\Delta_{25}}{\Delta_{24}}$$

$$t_{23} = 1 + t_{24}\frac{\Delta_{24}}{\Delta_{23}}$$

$$t_{31} = 1 - t_{23}\frac{\Delta_{23}}{\Delta_{31}}$$

$$t_{36} = 1 + t_{31}\frac{\Delta_{31}}{\Delta_{36}}$$

$$t_{35} = 1 + t_{36}\frac{\Delta_{36}}{\Delta_{35}}$$

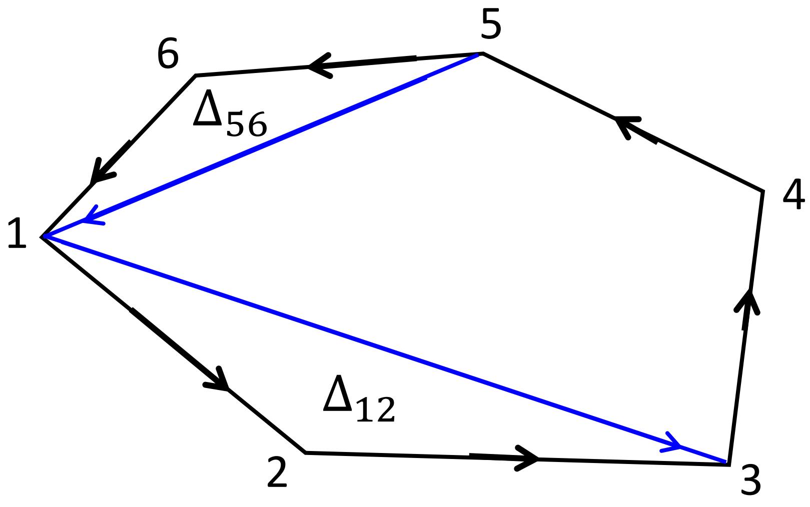
$$t_{34} = 1 + t_{35}\frac{\Delta_{35}}{\Delta_{34}}$$

$$t_{42} = 1 - t_{34}\frac{\Delta_{34}}{\Delta_{42}}$$

⋮

…

A curious property of the areas triangle

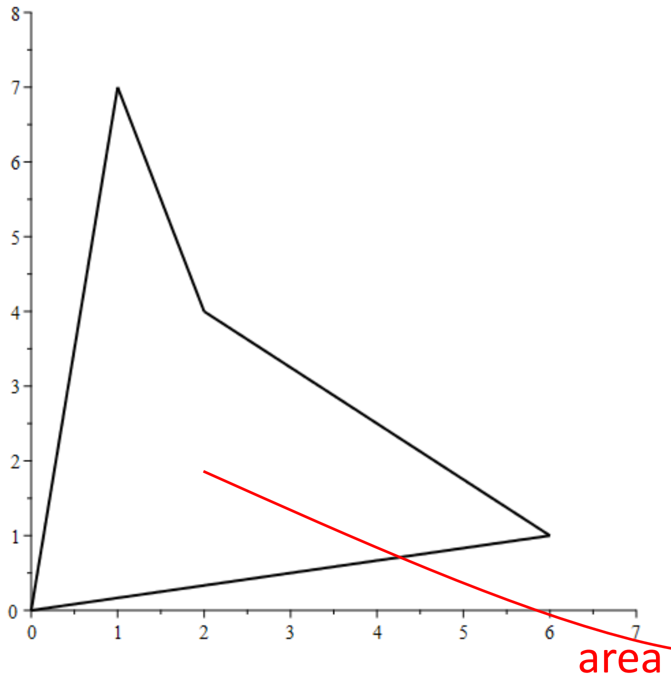


	1	2	3	4	5	6
1		Δ_{12}	Δ_{13}	Δ_{14}	Δ_{15}	
2			Δ_{23}	Δ_{24}	Δ_{25}	Δ_{26}
3	Δ_{31}			Δ_{34}	Δ_{35}	Δ_{36}
4	Δ_{41}	Δ_{42}			Δ_{45}	Δ_{46}
5	Δ_{51}	Δ_{52}	Δ_{53}			Δ_{56}
6	Δ_{61}	Δ_{62}	Δ_{63}	Δ_{64}		

Note negative area for triangle lying outside polygon

$$\begin{bmatrix} 0 & 11 & 5 & 0 \\ 0 & 0 & -\frac{9}{2} & \frac{41}{2} \\ 11 & 0 & 0 & 5 \\ \frac{41}{2} & -\frac{9}{2} & 0 & 0 \end{bmatrix}$$

For $n \geq 4$, it has zero determinant. In fact, all but three of its eigenvalues is zero. One of the nonzero eigenvalues is the area of the polygon.



Eigenvalues: $0, 16, -8 \pm \frac{i}{2}\sqrt{917}$

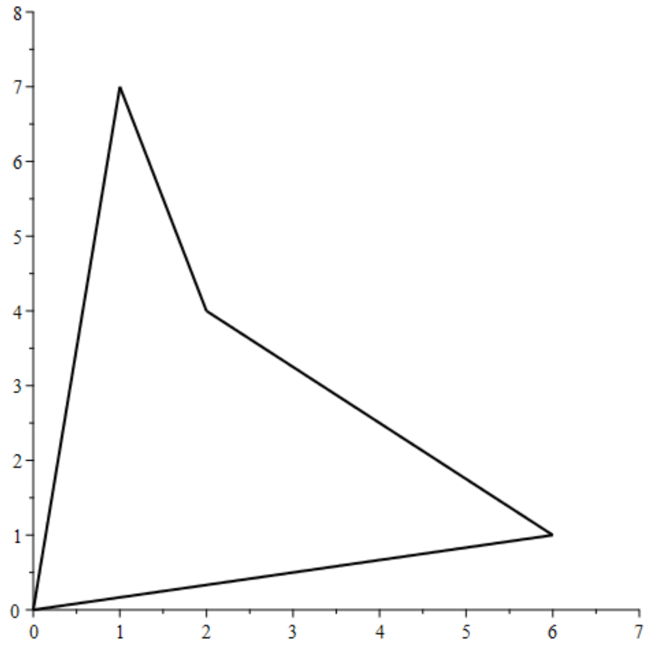
The characteristic polynomial

The characteristic polynomial for matrix A is the determinant of $A - qI$ where q is an indeterminate.

Its roots are the eigenvalues of the matrix:

$$\det(A - qI) = 0$$

$\Rightarrow$ $A - qI$ has rank less than n
$\Rightarrow$ the linear equations $(A - qI)\boldsymbol{x} = 0$ have a nontrivial solution vector (eigenvector) $\boldsymbol{x}$.



$$\begin{bmatrix} 0 & 11 & 5 & 0 \\ 0 & 0 & -\frac{9}{2} & \frac{41}{2} \\ 11 & 0 & 0 & 5 \\ \frac{41}{2} & -\frac{9}{2} & 0 & 0 \end{bmatrix}$$

$$\det(A - qI) = q^4 + \frac{149}{4} q^2 - 4692q$$

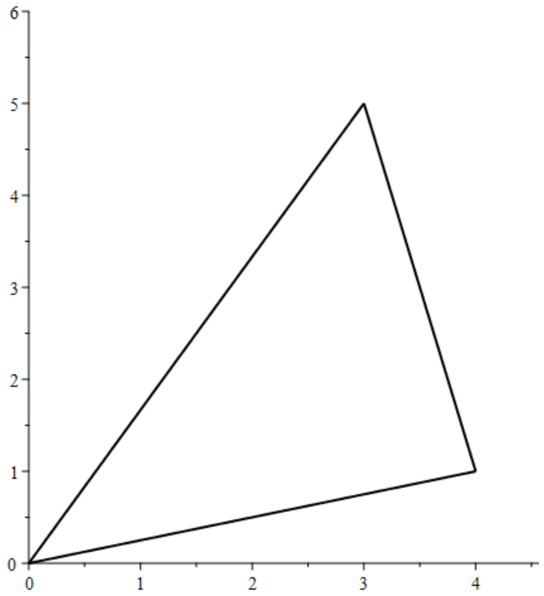
Eigenvalues (roots): $0, 16, -8 \pm \frac{i}{2}\sqrt{917}$

The characteristic polynomial puzzle

Characteristic polynomial of areas triangle for n-vertex polygon with area P is

$$\det(A - qI) = q^{n-3}(q - P)(q + P/2 \pm \alpha i)$$

What is α?

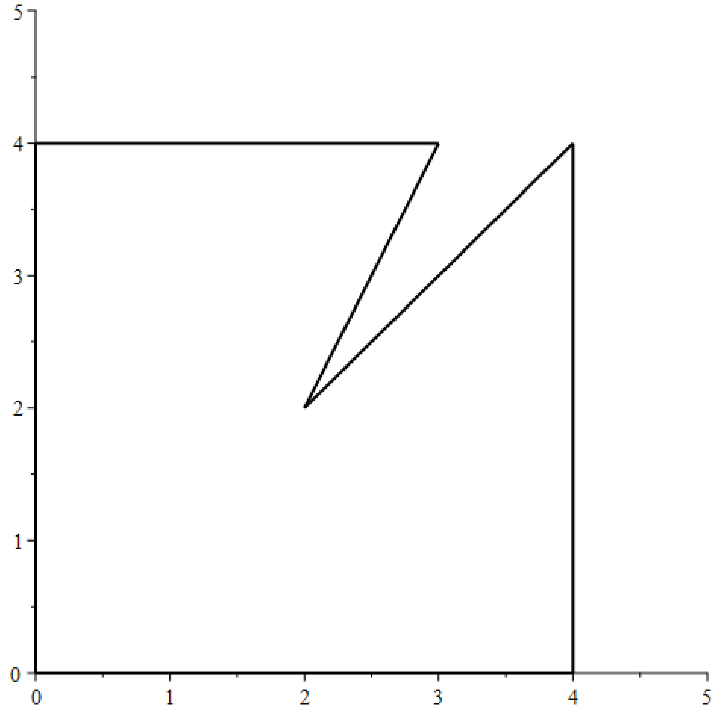


$$sw := \begin{bmatrix} 0 & \frac{17}{2} & 0 \\ 0 & 0 & \frac{17}{2} \\ \frac{17}{2} & 0 & 0 \end{bmatrix}$$

$$q^3 - \frac{4913}{8}$$

$$\frac{17}{2}, -\frac{17}{4} - \frac{17 I\sqrt{3}}{4}, -\frac{17}{4} + \frac{17 I\sqrt{3}}{4}$$

$$\left(q - \frac{17}{2}\right)\left(q + \frac{17}{4} \pm 17\frac{\sqrt{3}}{4}i\right)$$



$$sw := \begin{bmatrix} 0 & 8 & 0 & 1 & 6 & 0 \\ 0 & 0 & 4 & -3 & 6 & 8 \\ 8 & 0 & 0 & -1 & 0 & 8 \\ 4 & 4 & 0 & 0 & 3 & 4 \\ 8 & 2 & -1 & 0 & 0 & 6 \\ 8 & 8 & -4 & 3 & 0 & 0 \end{bmatrix}$$

$$q^6 - 96\,q^4 - 1935\,q^3$$

$$0, 0, 0, 15, -\frac{15}{2} - \frac{I\sqrt{291}}{2}, -\frac{15}{2} + \frac{I\sqrt{291}}{2}$$

$$q(q - 15)\left(q + \frac{15}{2} \pm \frac{\sqrt{291}}{2}i\right)$$

En passant, zero area triangles

We calculate the t_{ij} values by our feed-forward method:

$$t_{15} = 1 - \frac{P}{2\Delta_{15}}$$

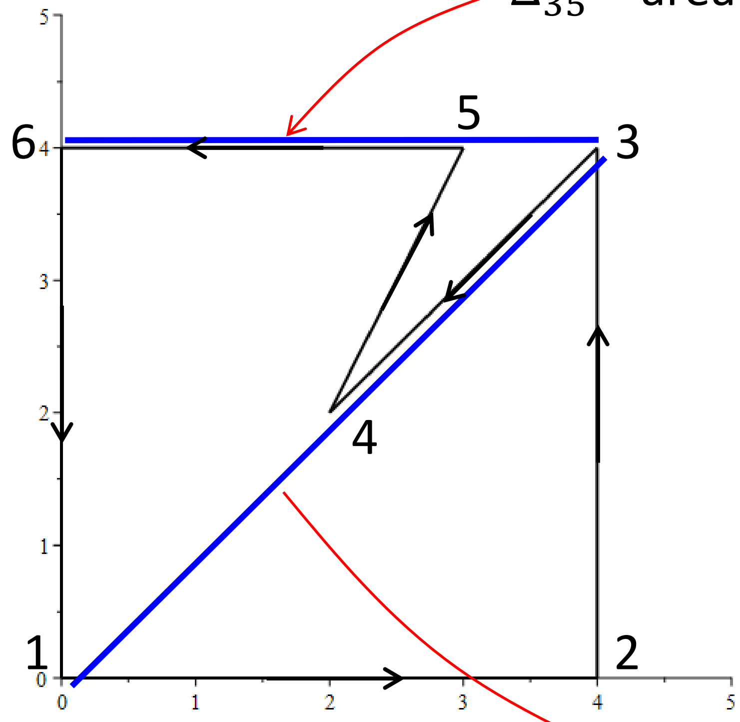
$$t_{14} = 1 + t_{15}\frac{\Delta_{15}}{\Delta_{14}}$$

$$t_{13} = 1 + t_{14}\frac{\Delta_{14}}{\Delta_{13}}$$

$$t_{12} = 1 + t_{13}\frac{\Delta_{13}}{\Delta_{12}}$$

What happens if Δ_{13}, say, is zero?

Δ_{35}= area of Δ 356 =0

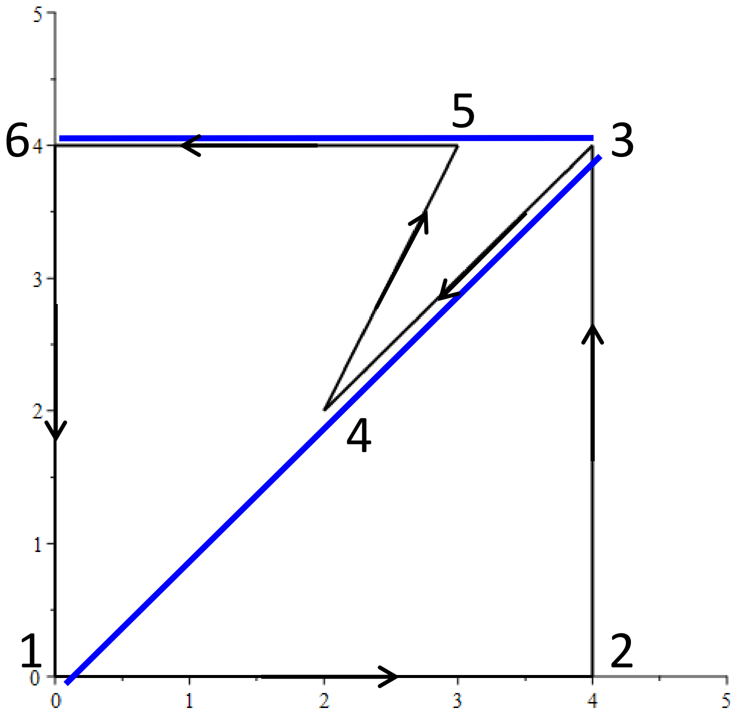


$$\begin{bmatrix} 0 & 8 & 0 & 1 & 6 & 0 \\ 0 & 0 & 4 & -3 & 6 & 8 \\ 8 & 0 & 0 & -1 & 0 & 8 \\ 4 & 4 & 0 & 0 & 3 & 4 \\ 8 & 2 & -1 & 0 & 0 & 6 \\ 8 & 8 & -4 & 3 & 0 & 0 \end{bmatrix}$$

Δ_{13}= area of Δ 134 =0

A trick

What happens if Δ_{13}, say, is zero?



$$t_{15} = 1 - \frac{P}{2\Delta_{15}}$$

$$t_{14} = 1 + t_{15}\frac{\Delta_{15}}{\Delta_{14}}$$

$$t_{13} = 1 + t_{14}\frac{\Delta_{14}}{x}$$

$$t_{12} = 1 + t_{13}\frac{x}{\Delta_{12}} = 1 + \left(1 + t_{14}\frac{\Delta_{14}}{x}\right)\frac{x}{\Delta_{12}}$$

$$= 1 + \frac{x}{\Delta_{12}} + t_{14}\frac{\Delta_{14}}{\Delta_{12}}$$

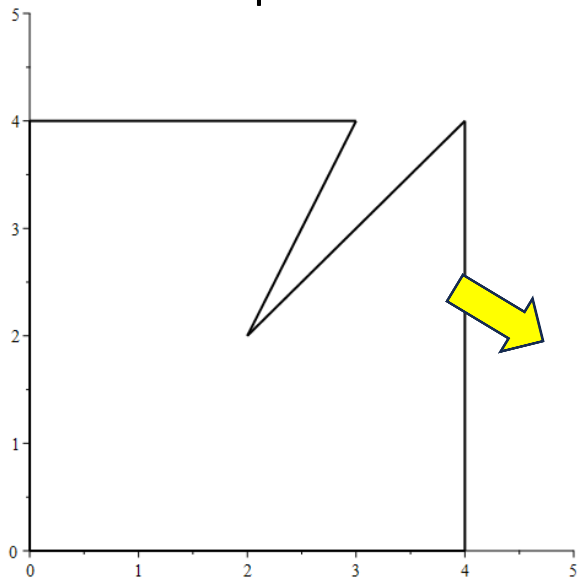
Now let $x \to 0$:

$$t_{12} = 1 + 0 + t_{14}\frac{\Delta_{14}}{\Delta_{12}} = 1 + t_{14}\frac{\Delta_{14}}{\Delta_{12}}$$

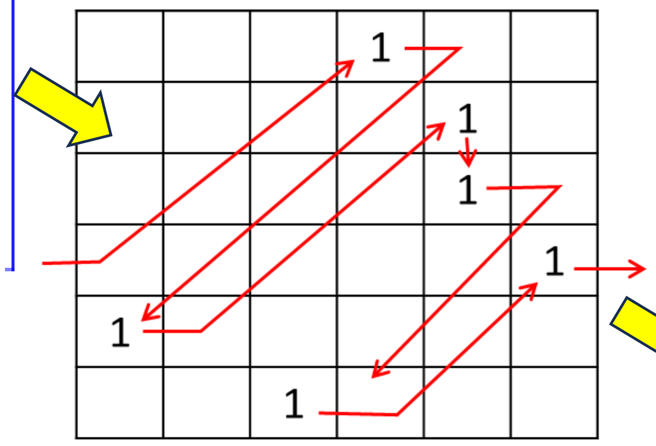
The effect is: we just ‘skip over’ zero entries in the areas matrix

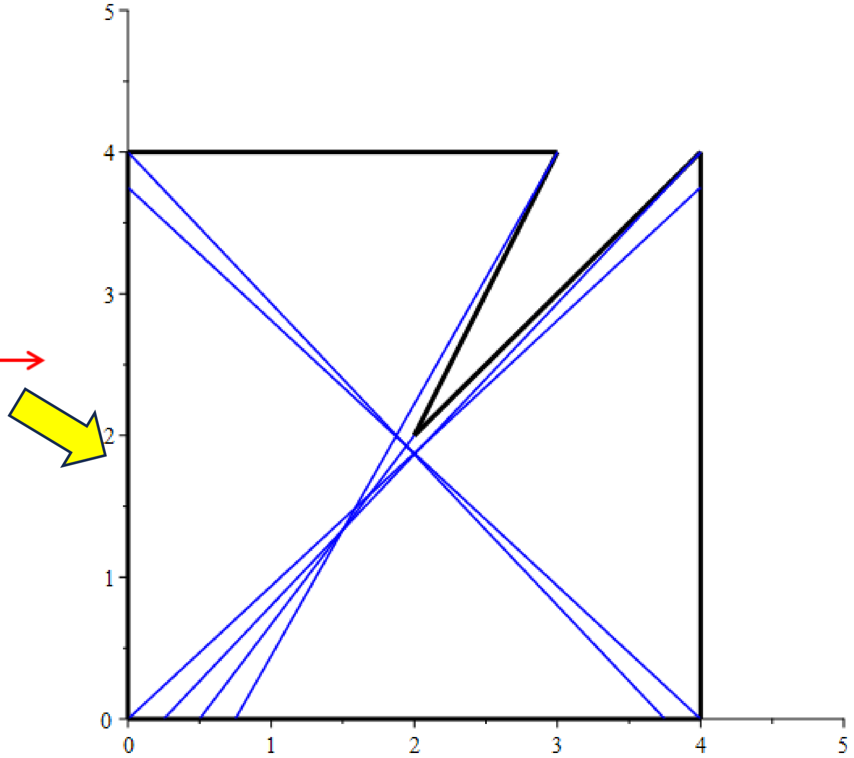
Plotting bisecting lines

We can now plot what I claim is a set of bisecting chords which, taken in the order traced in the toroidal matrix, define a partition of the polygon into sectors



$$\begin{bmatrix} 0 & 8 & 0 & 1 & 6 & 0 \\ 0 & 0 & 4 & -3 & 6 & 8 \\ 8 & 0 & 0 & -1 & 0 & 8 \\ 4 & 4 & 0 & 0 & 3 & 4 \\ 8 & 2 & -1 & 0 & 0 & 6 \\ 8 & 8 & -4 & 3 & 0 & 0 \end{bmatrix}$$





Easier examples to visualise