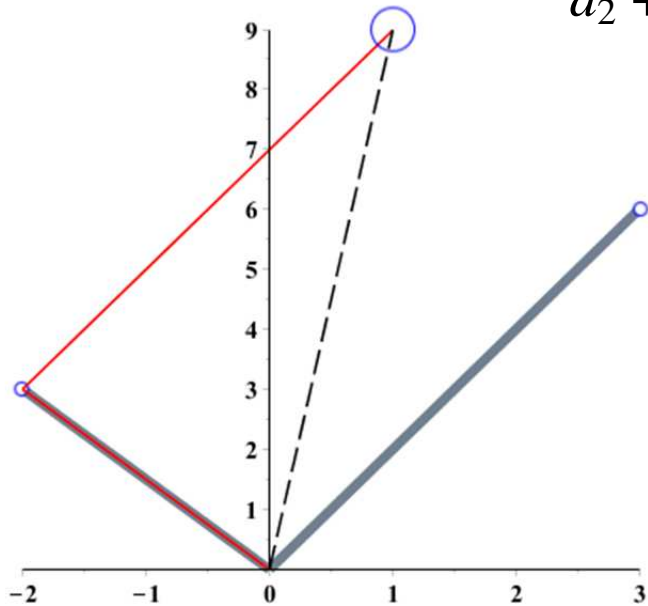




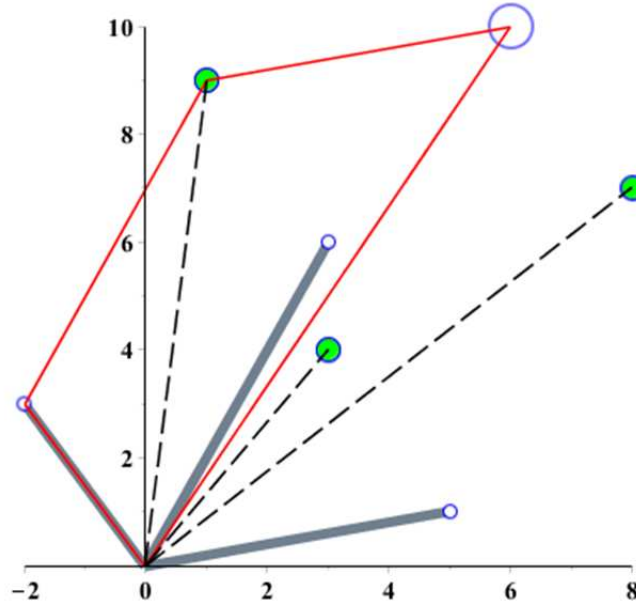
THEOREM OF THE DAY

A Generalised Hlawka Inequality Let x_1, x_2, \dots, x_n be a sequence of vectors in an inner product space (e.g. \mathbb{R}^m) with norm $\|\cdot\|$. Denote by d_k the sum of all norms of k -sums of these vectors, i.e. $d_k = \sum_{X \subseteq \{1, \dots, n\}, |X|=k} \|\sum_{i \in X} x_i\|$. Then, for $n \geq 3$,

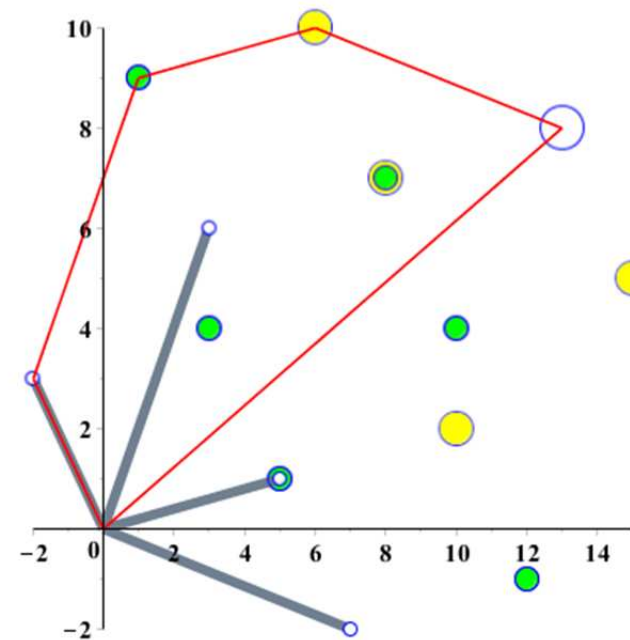
$$d_2 + \dots + d_{n-1} \leq (2^{n-2} - 1)(d_1 + d_n).$$



Triangle Inequality: $\|x + y\| \leq \|x\| + \|y\|$.



Quadrilateral (Hlawka) Inequality: $\|x+y\| + \|x+z\| + \|y+z\| \leq \|x\| + \|y\| + \|z\| + \|x+y+z\|$.



A pentagonal inequality: $d_2 + d_3 \leq 3(d_1 + d_4)$.

The Triangle Inequality is one of the defining properties of a norm; in this case the norm is that associated with an inner product which we take, for simplicity, to be defined as $\|x_i\| = \|(x_{i1}, x_{i2}, \dots, x_{im})\| = \sqrt{x_{i1}^2 + x_{i2}^2 + \dots + x_{im}^2}$. Depicted in the two-dimensional plot above left, the length of a vector sum (dashed line) is asserted to be no greater than the sum of the lengths of the constituent vectors. In our abbreviated notation, $d_2 \leq d_1$. Much less familiar, the Quadrilateral Inequality also holds, and this may be interpreted geometrically as shown above centre: the total length over all sums of pairs from three vectors (dashed lines) is not greater than the perimeter of the quadrilateral defined by the three vectors: $d_2 \leq d_1 + d_3$. And this latter inequality generalises: for four vectors (above right) we are totalling lengths over all pair sums (small solid green circles) and all three-sums (larger solid yellow circles); the result is not greater than three times the perimeter of the pentagon. (The dashed lines are omitted for clarity, in any case some vectors count twice, e.g. the point (8, 7) is both a two-sum and a three-sum, so $\sqrt{8^2 + 7^2}$ is twice a summand in the length total.)

In 1942 the Austrian mathematician Hans Hornich published an inequality concerned with offsetting the summands of two equal vector sums by a fixed unit vector. The Quadrilateral Inequality was a special case, attributed by Hornich to Edmund Hlawka. The generalisation given here is due to Dragomir Djoković and, independently, to Dorothy Manning Smiley and Malcolm Finley Smiley, who gave the geometric interpretation as illustrated above.

Web link: Włodzimierz Fechner's [paper](http://link.springer.com/journal/10/87/1/page/1) at link.springer.com/journal/10/87/1/page/1

Further reading: *The Cauchy-Schwarz Master Class: An Introduction to the Art of Mathematical Inequalities* by J. Michael Steele, Mathematical Association of America, 2004.