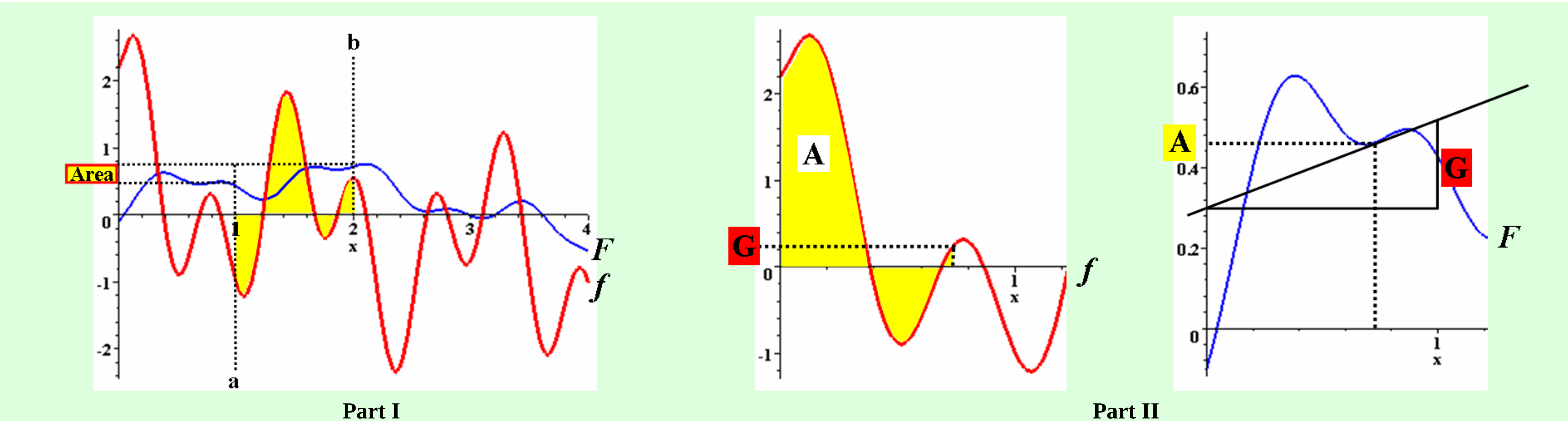
THEOREM OF THE DAY

The Fundamental Theorem of the Calculus

Part I: If f is continuous on the closed interval $[a,b]$ and $F(x)$ is a function for which $\frac{dF}{dx} = f(x)$ in that interval (F is an antiderivative *of f), then $\int_a^b f(x)dx = F(b) - F(a)$.*

Part II: An antiderivative for f exists: if f is continuous on an open interval around zero, and $F(x) = \int_0^x f(t)dt$, then $\frac{dF}{dx} = f(x)$, for every point in that interval.



Part I

Part II

Part I: No matter how complicated the continuous function f in the interval $[a,b]$, if we can find an antiderivative F, then calculating the shaded area is just a simple subtraction: $F(b) - F(a)$. The area under curve $f(x)$ from $x = a$ to $x = b$ (area below the horizontal axis counting as negative) can be identified with the *definite integral* $\int_a^b f(x)dx$ (it does no harm to think of $\int_a^b dx$ as a rather unusual notation for area).

Part II: And we can find an antiderivative F by measuring the area under f at every point away from zero. Unfortunately this will not normally give us a nice closed formula whereby area might be measured so neatly using Part I. Nevertheless, the duality between f and F that maps values G of the function f to gradients G on the curve F (values of dF/dx); and areas A under curve f (values of $\int f dx$) to values A of the function F, is one of the most fundamental in mathematics.

Part I can be traced to Nicole Oresme in the 14th century. In the 17th century, James Gregory, Isaac Barrow, Isaac Newton and Gottfried Leibniz were involved in the theorem's 'proto-calculus' formulation; then Poisson, Cauchy and Paul du Bois-Reymond developed its modern form in the 18th, with a thorough understanding of integration arriving only in the late 19th.

Web link: math.clarku.edu/~ma121/ has nice notes on Origins and on Proofs (scroll down to 'Syllabus'); more history at: "Teaching the Fundamental Theorem of the Calculus".

Further reading: *A Radical Approach to Lebesgue's Theory of Integration* by David M. Bressoud, CUP, 2008, chapter 1.