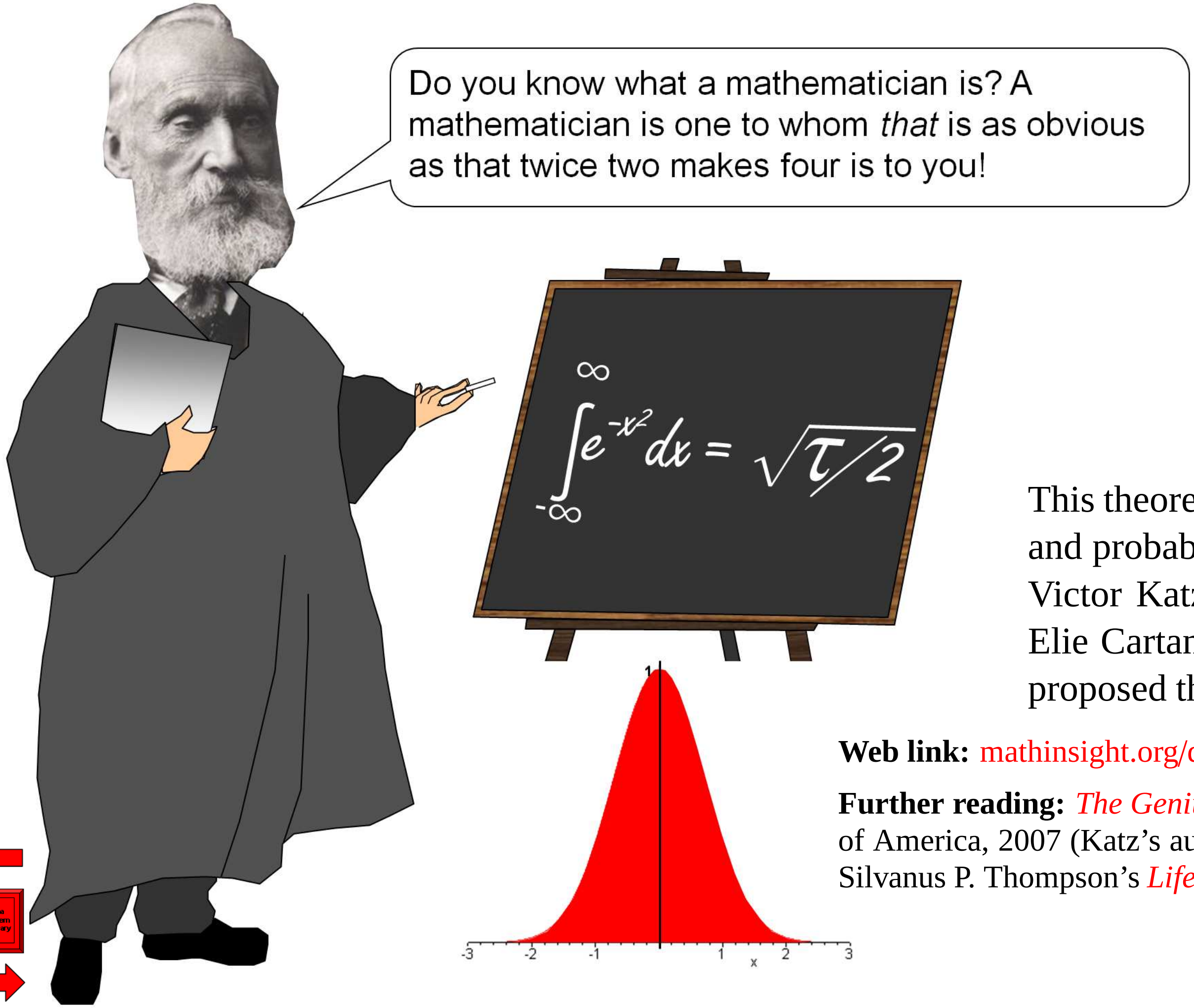
THEOREM OF THE DAY

The Change of Variables Theorem *Let A be a region in $\mathbb{R}^2$ expressed in coordinates x and y. Suppose that region B in $\mathbb{R}^2$, expressed in coordinates u and v, may be mapped onto A via a $1-1$ transformation T specified by continuously differentiable functions $x = p(u, v)$ and $y = q(u, v)$. Then for a continuous function f on A,*

$$\iint_A f\,dxdy = \iint_B f(p(u,v), q(u,v))|J_T(u,v)|dudv,$$

where $J_T(u,v) = \det\begin{pmatrix} \partial p/\partial u & \partial p/\partial v \\ \partial q/\partial u & \partial q/\partial v \end{pmatrix}$ *is the* Jacobian *of T.*

$\tau = 2\pi$ and e in another magical combination!



Lord Kelvin's integral on the blackboard offers a famous example of how today's theorem can dramatically simplify a problem. Let $I = \int_{-\infty}^{\infty} e^{-x^2}dx$. Then $I^2 = (\int_{-\infty}^{\infty} e^{-x^2}dx)(\int_{-\infty}^{\infty} e^{-y^2}dy) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}dxdy$. We propose to change variable by expressing the xy plane in polar coordinates, via the transformation $x = p(r,\theta) = r\cos\theta$, $y = q(r,\theta) = r\sin\theta$. Then the Jacobian is given by $\det\begin{pmatrix} \partial r\cos\theta/\partial r & \partial r\cos\theta/\partial\theta \\ \partial r\sin\theta/\partial r & \partial r\sin\theta/\partial\theta \end{pmatrix} = \det\begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix} = r$. Now, Change of Variables gives $I^2 = \int_0^{\tau}\int_0^{\infty} e^{-r^2(\cos^2\theta+\sin^2\theta)}r\,drd\theta = \int_0^{\tau}[-\frac{1}{2}e^{-r^2}]_0^{\infty}d\theta = \int_0^{\tau}\frac{1}{2}d\theta = \tau/2$.

This theorem, whose use is second nature to applied mathematicians and probability theorists, was surprisingly resistent to formal proof. Victor Katz attributes its first completely satisfactory treatment to Elie Cartan in the 1890s, over 125 years after Leonhard Euler first proposed the technique while inventing double integration.

Web link: mathinsight.org/double_integral_change_variables_introduction

Further reading: *The Genius of Euler* by William Dunham (ed.), the Mathematical Association of America, 2007 (Katz's authoritative paper is reproduced in part 2). The Kelvin quote is from Silvanus P. Thompson's *Life of Lord Kelvin, vol.2*, OUP.