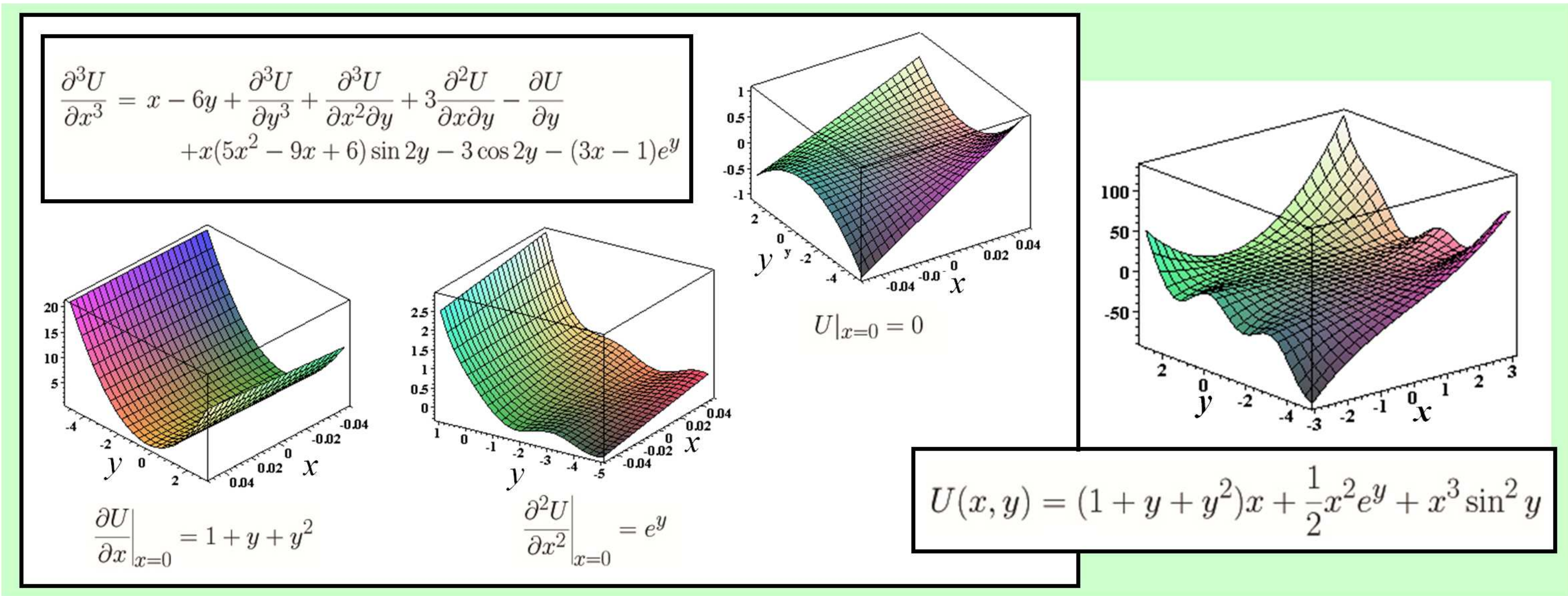
THEOREM OF THE DAY

The Cauchy-Kovalevskaya Theorem *Suppose that* $f^{(0)}, \ldots, f^{(k-1)}$, $k \geq 1$, *are entire functions on* $\mathbb{C}^n$, $n \geq 1$, *and* h *is an entire function on* $\mathbb{C}^{n+1}$. *Let* $U(x, \underline{y})$ *be an unknown function on* $\mathbb{C}^{n+1}$, $\underline{y} = (y_1, \ldots, y_n)$. *Then the* k*-th order partial differential equation* $D_x^k U = ax + b\underline{y} + \sum_{0 \leq i < k,\ i + |\underline{j}| \leq k} c_{i,\underline{j}} D_x^i D_{\underline{y}}^{\underline{j}} U + h(x, \underline{y})$, *with* $\underline{j}$ *indexing the entries of* $\underline{y}$, *subject to initial conditions* $D_x^i U|_{x=0} = f^{(i)}(\underline{y})$, $0 \leq i \leq k-1$, *has an entire solution for* U.



The partial differential equation (PDE) in this example, would be written $D_x^3 U = x - 6y + D_y^3 U + D_x^2 D_y^1 U + 3D_x D_y U - D_y U + h(x, y)$, in the notation of the theorem, with y being a single variable. The function h is a sum of products of polynomials, sines, cosines, and the exponential function, and this ensures that it is *entire*: complex-differentiable everywhere, a formalisation of the idea of being 'well-behaved'. The initial condition functions, graphed above-left for a narrow strip about the line $x = 0$, similarly qualify as entire, as does the solution $U(x, y)$ graphed on the right.

In 1874, Sofia Kovalevskaya completed three papers, each deemed by the great Weierstrass to be individually worthy of a doctorate. Her theorem on PDEs massively generalised previous results of Cauchy on convergence of power series solutions and applies far beyond the version stated here, to systems of (nonlinear) PDEs and requiring only locally holomorphic functions.

Web link: mathshistory.st-andrews.ac.uk/Projects/Ellison/chapter-5/. And Garry J. Tee's much-cited article on Kovalevskaya (3.7MB): www.theoremoftheday.org/Docs/KovalevskayaTee/KovalevskayaTee.pdf (courtesy G.J. Tee & the Mathematical Chronicle Committee).

Further reading: *Advanced Real Analysis* by Anthony W. Knapp, Birkhauser, 2005.