THEOREM OF THE DAY

Sylvester's Law of Inertia *If* $Q(x_1,\ldots x_n) = a_{11}x_1^2 + \ldots + a_{nn}x_n^2 + 2\sum_{i<j} a_{ij}x_ix_j$ *is any quadratic form with real coefficients then there exists a linear transformation*

$$\begin{aligned} y_1 &= \gamma_{11}x_1 + \ldots + \gamma_{1n}x_n, \\ y_2 &= \gamma_{21}x_1 + \ldots + \gamma_{2n}x_n, \\ &\ldots \\ y_n &= \gamma_{n1}x_1 + \ldots + \gamma_{nn}x_n, \end{aligned}$$

and coefficients $b_1, b_2, \ldots, b_n \in \{0, 1, -1\}$, *such that* $Q(x_1,\ldots x_n) = b_1y_1^2 + \ldots b_ny_n^2$. *Moreover the matrix* (γ_{ij}) *of coefficients can always be chosen to be nonsingular and then the values* $\mathrm{rank}(Q) = \sum |b_i|$ *and* $\mathrm{signature}(Q) = \sum b_i$ *are uniquely determined, i.e. are invariants of* Q.

$$Q(x_1, x_2, x_3) = 3x_1^2 + 10x_2^2 - 5x_3^2 + 12x_1x_2 - 6x_1x_3 - 4x_2x_3$$

$$A: \begin{pmatrix} 3 & 6 & -3 \\ 6 & 10 & -2 \\ -3 & -2 & -5 \end{pmatrix}$$

$$S: \begin{aligned} y_1 &= \tfrac{\sqrt{10}}{5}(3x_1 + 5x_2 - x_3) \\ y_2 &= \tfrac{1}{14}(-3x_1 + 2x_2 + x_3) \\ y_3 &= \tfrac{\sqrt{15}}{5}(x_1 + 3x_3) \end{aligned} \qquad \begin{pmatrix} \frac{3}{5}\sqrt{10} & \sqrt{10} & -\frac{1}{5}\sqrt{10} \\ -\frac{3}{14} & \frac{1}{7} & \frac{1}{14} \\ \frac{1}{5}\sqrt{15} & 0 & \frac{3}{5}\sqrt{15} \end{pmatrix}$$

$$Q(x_1, x_2, x_3) = y_1^2 - y_3^2$$

$$(S^{-1})^T A S^{-1}: \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$T: \begin{aligned} y_1 &= \sqrt{3}(x_1 + 2x_2 - x_3) \\ y_2 &= \sqrt{2}(x_2 - 2x_3) \\ y_3 &= x_3 \end{aligned} \qquad \begin{pmatrix} \sqrt{3} & 2\sqrt{3} & -\sqrt{3} \\ 0 & \sqrt{2} & -2\sqrt{2} \\ 0 & 0 & 1 \end{pmatrix}$$

$$Q(x_1, x_2, x_3) = y_1^2 - y_2^2$$

$$(T^{-1})^T A T^{-1}: \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

A nonsingular transformation can be thought of as acting on the real symmetric matrix A representing a quadratic form Q, via X^TAX, where X is the inverse of the matrix of coefficients of the transformation. Here we see two different transformations of the quadratic form Q both giving rank $= |1| + |-1| = 2$ and signature $= 1 + -1 = 0$.

Joseph Sylvester published a new proof of this reduction of quadratic forms in 1852, citing previous proofs by Cauchy, Jacobi and Borchardt but drawing particular attention to its invariant nature: "a law to which my view of the physical meaning of quantity of matter inclines me, upon the ground of analogy, to give the name of the Law of Inertia for Quadratic Forms".

Web link: www.maths.ed.ac.uk/~aar/sylv

Further reading: *James Joseph Sylvester: Jewish Mathematician in a Victorian World* by Karen H. Parshall, The Johns Hopkins University Press, 2006.