A translation into English of the opening paragraphs of Schur, J., "Zur Theorie der vertauschbaren Matrizen", *Journal für die reine und angewandte Mathematik*, vol. 130, 1905, pp. 66–76. The original can be read in facsimile at the Göttinger Digitaisierungszentrum gdz.sub.uni-goettingen.de/id/PPN243919689_0130. Note that Issai Schur often gave his initial as 'J' in publications.

On the theory of commuting matrices

by Mr J. Schur, Berlin

As is well known, among the matrices of degree n

$$a_{\chi\lambda} \qquad\qquad (\chi\lambda = 1, 2, \ldots, n),$$

n^2 linearly independent matrices can be specified in an infinite number of ways. Any such collection $A_0, A_1, \ldots, A_{n^2-1}$, then has the property that any other matrix A of degree n can be written in the form

$$A = c_0 A_0 + c_1 A_1 + \ldots + c_{n^2-1} A_{n^2-1}.$$

Since for $n > 1$ matrix multipication is not in general commutative, there cannot be a system of n^2 linearly independent matrices of the nth degree which are pairwise commutative. Thus the question arises of determining the greatest possible number of linearly independent, mutually commuting matrices of the nth degree. Let the number we are looking for be denoted by $\nu_n + 1$. If then

$$A_0, A_1, \ldots, A_{\nu_n}$$

are $\nu_n + 1$ commuting matrices which are linearly independent, then clearly each of the products $A_\beta A_\gamma$ must be represented in the form

$$A_\beta A_\gamma = \sum_{\alpha=0}^{\nu_n} c_{\alpha\beta\gamma} A_\alpha,$$

where the $c_{\alpha\beta\gamma}$ are certain constants. The collection of all matrices

$$x_0 A_0 + x_1 A_1 + \ldots + x_{\nu_n} A_{\nu_n}$$

with arbitrary coefficients x_ν therefore forms a commutative algebra of order $\nu_n + 1$ according to the definition of Mr Frobenius[1]. The problem posed above can therefore also be formulated as follows: "Which is the greatest possible order of a commutative algebra of order $\nu_n + 1$.

We prove the following:

I: The order of a commutative algebra of degree n matrices is at most $\left[\frac{n^2}{4}\right] + 1$.

Here, as usual, $\left[\frac{n^2}{4}\right]$ means the largest integer that does not exceed $\frac{n^2}{4}$, i.e. for even n the number $\frac{n^2}{4}$ and for odd n, the number $\frac{n^2-1}{4}$.

References

[1] Frobenius, "Theorie der hyperkomplexen Grössen", *Sitzungsberichte der Berlin Akademie*, 1903, S. 504.