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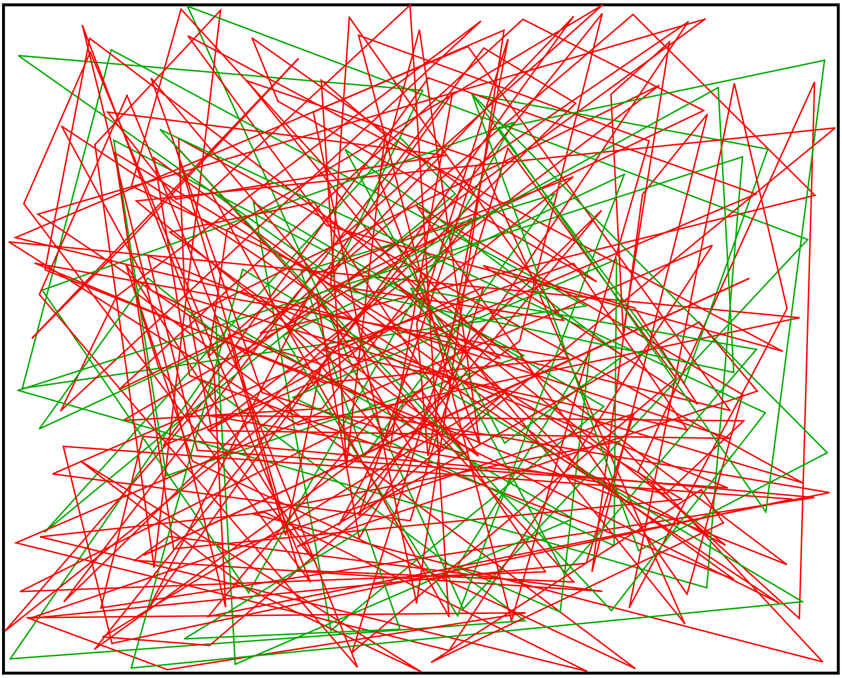

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Editor – *Tony Forbes*

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Falling ladder, hidden contraption

J. M. Selig

A colleague told me about a conversation she had had with a Hungarian mathematician. This guy wanted to illustrate what he thought was a 'Hungarian' approach to mathematics as opposed to what he thought of as an 'Anglo-Saxon' style.

Apparently he asked my colleague to solve the following problem: A ladder rests against a vertical wall. Suddenly, for undisclosed reasons, the friction between the ladder and the wall and floor disappears, so the ladder falls. As it falls it never loses contact with the wall or the floor. What curve does the centre point of the ladder trace as it falls?

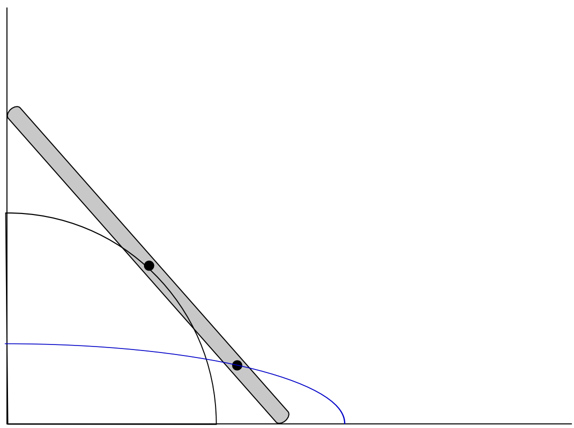


Figure 1: A ladder falling down a wall.

My colleague dutifully drew a sketch, probably something like Figure 1, wrote down some equations and solved the problem. "Aha! You see. A Hungarian mathematician would have immediately spotted that the figure looks like the theorem in Euclidean geometry that says that the angle in a semicircle is a right-angle and so the path of the point is obviously a circle."

I would have approached this problem completely differently. To begin with, I would have tried to annoy the guy by asking for the shape of the trajectory of any other point on the ladder. Its an ellipse; I'll explain later why I know this. First I want to say something about rigid-body displacements.

1 Planar Rigid-Body Displacements

To my mind the ladder is not just a line but a rigid-body. It only moves in a vertical plane, so we only need to consider rigid-body displacements in the plane.

Rigid-body displacements are displacement that preserve the Euclidean distance between points. Examples are rotations, translations and reflections. In fact, it can be shown that all rigid-body displacements are combinations of these three. Reflections will not be considered here, since these cannot result from any physical operation we can do to a body, such as a ladder. When we ignore the reflections we should really talk of proper rigid-body displacements, but the qualifier, 'proper' will usually be dropped in the interests of brevity.

A rotation about the origin can be represented by a 2×2 matrix,

$$R = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix},$$

where θ is the rotation angle measured positive in the anti-clockwise direction. Given a point p in the plane with coordinates (x, y), the effect of the rotation is given by the product

$$R\vec{p} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \end{pmatrix}.$$

In this representation a translation by a vector $\vec{t} = (t_x, t_y)$ is given by simply adding the translation vector to the position vector of the point $\vec{p} + \vec{t}$. A slightly neater way to represent rotations and translations is to use 3×3 matrices. Now, a rotation about the origin is given by the matrix

$$\tilde{R} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and a translation by} \quad \tilde{T} = \begin{pmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{pmatrix}.$$

In order to represent the action of these displacements on points in the plane we need to extend the position vectors to 3-component vectors,

$$\tilde{p} = \begin{pmatrix} x \\ y \\ 1 \end{pmatrix};$$

the final 1 here is not the z-component of anything.

The actions of the displacements on points are given by the matrix-vector multiplications for the rotation,

$$\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \longmapsto \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} x\cos\theta - y\sin\theta \\ y\sin\theta + x\cos\theta \\ 1 \end{pmatrix}.$$

The action of a translation is

$$\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \longmapsto \begin{pmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} x + t_x \\ y + t_y \\ 1 \end{pmatrix}.$$

As mentioned above, all proper rigid-body displacements can be thought of as compositions of rotations and translations. It is not difficult to see from the above that such displacements can be represented by products of the 3×3 rotation and translation matrices described. These displacements do not commute; the order that they are performed in is important. When multiplying these matrices the first operation is the rightmost matrix. So, if we want to perform a rotation by θ followed by a translation by $\vec{t}$, the matrix representing the overall displacement will be

$$\begin{pmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta & t_x \\ \sin\theta & \cos\theta & t_y \\ 0 & 0 & 1 \end{pmatrix}.$$

Performing the operation in the other order would give

$$\begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta & t_x\cos\theta - t_y\sin\theta \\ \sin\theta & \cos\theta & t_x\sin\theta + t_y\cos\theta \\ 0 & 0 & 1 \end{pmatrix}.$$

2 Motion of the Ladder

The motion of the ladder as it falls can be thought of as a combination of two simple motions. If the ladder is initially vertical then we can think of it rotating about the origin. Of course, this motion is only imagined since it would put the ladder inside the wall. This rotation must be combined with a translation in the x-direction, the magnitude of this translation must keep the top of the ladder on the vertical wall. So, if the rotation of the ladder about the origin is given by

$$\tilde{R} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}; \text{ the translation will be } \tilde{T} = \begin{pmatrix} 1 & 0 & 2l\sin\theta \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where $2l$ is the length of the ladder. The motion of the ladder is thus the combination

$$\begin{pmatrix}1 & 0 & 2l\sin\theta\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}\cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{pmatrix} = \begin{pmatrix}\cos\theta & -\sin\theta & 2l\sin\theta\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{pmatrix}.$$

The mid-point of the ladder is initially located at the point $(0,\, l)$ and hence its trajectory will be given by

$$\begin{pmatrix}\cos\theta & -\sin\theta & 2l\sin\theta\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}0\\ l\\ 1\end{pmatrix} = \begin{pmatrix}l\sin\theta\\ l\cos\theta\\ 1\end{pmatrix}.$$

This is clearly a circle centred at the origin with radius l. The direction in which the circle is traced can be seen to be clockwise. What about other points on the ladder? What about a point that is initially at position $(0,\, h)$? We get

$$\begin{pmatrix}\cos\theta & -\sin\theta & 2l\sin\theta\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}0\\ h\\ 1\end{pmatrix} = \begin{pmatrix}(2l-h)\sin\theta\\ h\cos\theta\\ 1\end{pmatrix}.$$

This is an ellipse satisfying the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

where $a = (2l - h)$ and $b = h$. The eccentricity, e, of this ellipse is given by

$$e = \sqrt{1 - b^2/a^2} \ \text{ if } \ a > b \quad \text{and} \quad e = \sqrt{1 - a^2/b^2} \ \text{ if } \ b > a.$$

This is so that $0 < e < 1$ unless the ellipse is actually a circle, in which case $e = 0$. If the point we are considering is less than half-way up the ladder, $h < l$, so that $(2l - h) > h$ and the eccentricity of the ellipse is

$$e = \frac{2}{2(l/h) - 1}\sqrt{(l/h)^2 - (l/h)}.$$

If our point is between the mid-point and the top of the ladder, then $h > (2l - h)$. In this case the eccentricity of the ellipse will be

$$e = 2\sqrt{(l/h) - (l/h)^2}.$$

More generally, we can imagine attaching pieces to the ladder and asking about paths of points on these extensions. Suppose (x_0, y_0) are the initial coordinates of some point on this enlarged ladder. Its trajectory as the ladder falls is given by

$$\begin{pmatrix} \cos\theta & -\sin\theta & 2l\sin\theta \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \\ 1 \end{pmatrix} = \begin{pmatrix} x_0\cos\theta + (2l - y_0)\sin\theta \\ x_0\sin\theta + y_0\cos\theta \\ 1 \end{pmatrix}.$$

Writing

$$x = x_0\cos\theta + (2l - y_0)\sin\theta \quad \text{and} \quad y = x_0\sin\theta + y_0\cos\theta$$

for general points on the trajectory, we can see that

$$\cos\theta = \frac{xx_0 + y(y_0 - 2l)}{x_0^2 + y_0^2 - 2ly_0}, \qquad \sin\theta = \frac{yx_0 - xy_0}{x_0^2 + y_0^2 - 2ly_0}.$$

Hence, as the ladder falls, the point follows the curve given by the equation

$$\big(xx_0 + y(y_0 - 2l)\big)^2 + \big(yx_0 - xy_0\big)^2 = \big(x_0^2 + y_0^2 - 2ly_0\big)^2.$$

This is the equation of an ellipse centred at the origin but where the semi-major axis is not aligned with the x- or y-axis.

3 The Elliptic Trammel

Suppose you want to draw an ellipse, how would you do that? Most of us are taught at school that you can draw an ellipse by attaching a piece of string to a pair of fixed points, drawing pins (thumb tacks), for example. Then looping the string around a pencil and drawing the curve with the string held tight. Many of us have even tried this and can therefore confirm that it is a terrible way to draw an ellipse—the string stretches and the pencil wobbles. The point is not really that you can draw a nice ellipse like this but it shows you that the ellipse is the curve where the sum of the distances to a pair of fixed points is constant.

But suppose you really did want to draw an ellipse. Say you wanted to carve a elliptical shaped arch out of a block of solid stone to put above a window or door. Or maybe you are making a technical drawing in the days before CAD. How would you really draw an ellipse? The answer is: use an ellipsograph. This was a device based on the trammel of Archimedes. According to *Wikipedia* this is a mechanism that: ‘consists of two shuttles which are confined ('trammeled') to perpendicular channels or rails and a rod which is attached to the shuttles by pivots at fixed positions along the rod.' See [1]. Examples of these devices used to be on display at the Science Museum in London, see [2].

The mechanism has many other names including the 'do nothing machine', presumably because when people came across this machine more recently they were unaware or indifferent to its uses.

This is why I knew that the paths of other points on the ladder traced ellipses. Anyone familiar with the kinematics of planar mechanisms would have immediately recognised the motion of the ladder as the same as the motion of the moving bar in the elliptic trammel!

Yet another view of the geometry of this device can be seen in the YouTube video [3].

References

[1] Trammel of Archimedes, *Wikipedia*, `https://en.wikipedia.org/wiki/Trammel_of_Archimedes`, last accessed 29/3/2024.

[2] Science Museum London, `https://collection.sciencemuseumgroup.org.uk/objects/co60175/elliptic-trammels-curve-drawing-instruments-elliptical-trammels`, last accessed 29/3/2024.

[3] Burkard Polster (Mathologer), Secrets of the NOTHING GRINDER (2018), `https://www.youtube.com/watch?v=7Fn-26Jmi5E`, last accessed 24/3/2024.

Problem 319.1 – Sum

Tony Forbes

For positive integer r, show that

$$\frac{1}{r^2+(r+1)^2}+\frac{1}{(3r+1)^2+(3r+2)^2}+\frac{1}{(5r+2)^2+(5r+3)^2}+\dots = \frac{\pi}{4r+2}\tanh\frac{\pi}{4r+2}.$$

Solution 317.3 – Eight triangles

Denote the area of a triangle with vertices X, Y, Z by $\triangle(X,Y,Z)$. (i) A circle has the six points A, B, C, D, E, F in that order on its circumference. Show that

$$\begin{aligned}&\triangle(A,B,C)\triangle(D,E,F) - \triangle(A,B,D)\triangle(C,E,F)\\ &+ \triangle(A,C,D)\triangle(B,E,F) - \triangle(B,C,D)\triangle(A,E,F) \;=\; 0.\end{aligned}$$

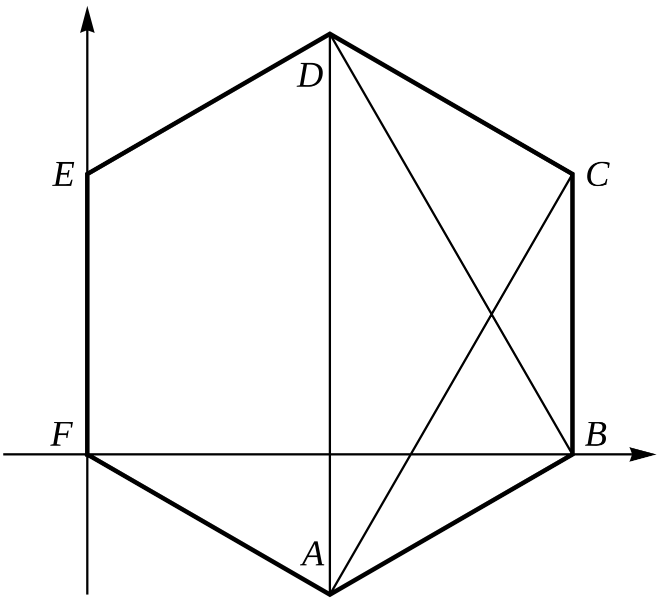
More generally, show that this holds for any convex shape.
(ii) Even more generally, choose any six points in the plane, A, B, C, D, E, F. Show that

$$\prod_{\epsilon_1,\epsilon_2,\epsilon_3=\pm 1} \Big(\triangle(A,B,C)\triangle(D,E,F) + \epsilon_1\,\triangle(A,B,D)\triangle(C,E,F) + \epsilon_2\,\triangle(A,C,D)\triangle(B,E,F) + \epsilon_3\,\triangle(B,C,D)\triangle(A,E,F)\Big) \;=\; 0,$$

or find a counter-example.

Robin Whitty

(i) We illustrate the problem in the following plot.



The problem proposes that we take a weighted sum of the triangle areas ABC and ACD and subtract off a weighted sum of the areas of ABD and BCD. If all the weights were unity, the result would automatically be zero because ABC and ACD form a quadrilateral and ABD and BCD will subtract off the same quadrilateral. But the weights are areas of different triangles on base edge EF. So it seems surprising that the zero sum should be preserved.

We are free to position the triangle edge EF on the vertical axis and to scale our configuration so that the length of EF is 2. Then the weight given to triangle ABC is precisely the height of triangle EFD, and so on: in the weighted sum, the weight of each triangle on the points A, B, C, D is the horizontal coordinate of the point excluded from it. If we take point A to have coordinates (x_A, y_A), etc, then we can restate the problem, reordering the terms slightly, as asking for a proof that

$$x_A\Delta(BCD) - x_B\Delta(ACD) + x_C\Delta(ABD) - x_D\Delta(ABC) \;=\; 0. \qquad (1)$$

We may collect the coordinates of A, B, C, D into a 3×4 matrix:

$$X \;=\; \begin{pmatrix} 1 & 1 & 1 & 1 \\ x_A & x_B & x_C & x_D \\ y_A & y_B & y_C & y_D \end{pmatrix}.$$

The determinant of any 3×3 submatrix of X is twice the area of the triangle formed on the corresponding points, taken in anticlockwise order. (In fact this generalises to simplexes of any dimension, and the version for tetrahedral volume was featured in Problem 316.6 in a preceding issue of M500.)

And now we may see that various zero weighted sums of areas come directly from Cramer's Rule for solving linear equations (M500, issue 262, page 7, if you have your back numbers to hand).

It is convenient to have a square matrix, so add an initial row of zeros to matrix X to give a 4×4 matrix W. Write $M_{i,j}$ for the determinant of the submatrix obtained from W by deleting the i-th row and j-th column and write C for the 4×4 matrix whose ij-th entry is $(-1)^{i+j}M_{i,j}$. Then Cramer's Rule says that multiplying W by the transpose of C gives the identity matrix multiplied by the determinant of W. For our matrix W, this determinant is zero, so we can write:

$$WC^{\mathrm{T}} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ x_A & x_B & x_C & x_D \\ y_A & y_B & y_C & y_D \end{pmatrix} \times \begin{pmatrix} M_{1,1} & & \\ -M_{1,2} & \dots & \\ M_{1,3} & & \\ -M_{1,4} & & \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \qquad (2)$$

In the first column of zeros in the right-hand side matrix, the entry in row 2 tells us that the weighted area sum with all weights unity is zero:

$$1 \times \Delta(BCD) - 1 \times \Delta(ACD) + 1 \times \Delta(ABD) - 1 \times \Delta(ABC) = 0;$$

the entry in row 3 is precisely equation (1); the entry in row 4 tells us that zero is preserved if the weights are the vertical coordinates of the 'missing' points instead of the horizontal coordinates.

(ii) The Cramer's Rule calculation supplies a positive answer to part (ii) of this problem. If the points in the above figure are ordered differently then two things happen: geometrically, the pairings of triangles forming the same quadrilateral need not be the same as before; and algebraically, the 3×3 determinants in matrix X may become negative (the triangle points are being taken in clockwise order). Nevertheless, equation (2) still holds and allows us to read off the values of the ϵ_i which supply a zero factor for the product.

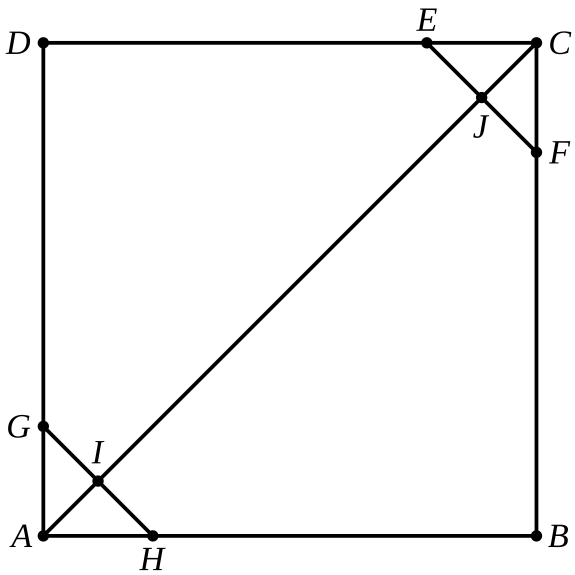
In fact, the same applies to simplexes of any dimension: the volumes of the five tetrahedra formed on five points in three dimensions also form a zero alternating sum, for instance. Again, however, the volumes are 'signed' according to the orientations of the tetrahedra. This is why the determinant formula for tetrahedral volume applies the absolute value function (and this is what is played upon in Problem 316.6 in M500). Again the product formulation in part (ii) of the problem offers an appropriate way of saying 'there is a choice of signs for which the sum of the volumes is zero'.

Solution 315.1 – Rectangles in a square

How many $1 \times (n+1)$ rectangles can you fit in an $n \times n$ square?

Obviously fitting any at all might be a bit difficult when $n = 1$ or 2. But as n increases the difference between n and $\sqrt{2}\,n$ becomes more and more significant.

Ted Gore



Let $CJ = AI = x$. Let $IJ = y$. Let $y = m(n+1)$, where m is the number of rectangles of length $n+1$ that can be placed sequentially along IJ. Let $EF = GH = k$, where k is the number of rectangles of width 1 that can be placed side by side along IJ.

We have $\cot(\pi/4) = 1 = 2x/k$, so that $x = k/2$. Then

$$\sqrt{2}n \;=\; y + 2x \;=\; m(n+1) + k$$

and

$$n \;=\; \frac{m+k}{\sqrt{2} - m}.$$

But n cannot be negative; so m cannot be greater than 1. When $m = 1$

and $k = 1$ we get

$$n = \frac{2}{\sqrt{2}-1} = 4.8284.$$

We can, however, fit k rectangles of width 1 side by side. For example $k = 2$ gives

$$n = \frac{3}{\sqrt{2}-1} = 7.2426$$

and $k = 3$ gives

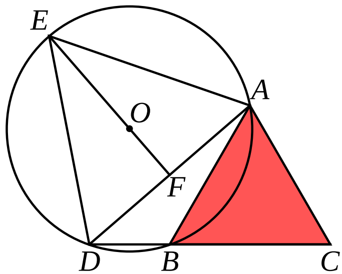
$$n = \frac{4}{\sqrt{2}-1} = 9.6569.$$

Solution 314.4 – A triangle and a circle

A circle that passes through A and B of equilateral triangle ABC meets BC at D.

The length of $|AD|$ is 1.

What's the area of the circle?



Ted Gore

Construct the perpendicular bisector of AD. The bisector passes through O, the centre of the circle. Extend it to meet the circle in the point E.

Then $ABDE$ is a cyclic quadrilateral in which the angles ABD and DEA are supplementary. Since $\triangle ABC$ is equilateral, $\angle ABC = \pi/3$ and $\angle ABD = 2\pi/3$. Thus $\angle DEA = \pi/3$ and $\angle DOA = 2\pi/3$ by the Central Angle Theorem.

Let F be the mid point of DA. We have $\angle FOA = \pi/3$ and $OA = r$, where r is the radius of the circle. From this we get

$$r = \frac{1}{2\sin(\pi/3)} = \frac{1}{\sqrt{3}} = 0.57735026919,$$

and the area of the circle is $\pi/3 = 1.0471975512$.

Solution 316.7 – Cubes

A $5\times5\times5$ cube is partitioned into 125 $1\times1\times1$ subcubes of which five are coloured red. The other 120 are coloured something other than red. Each $1\times5\times5$ slice of the cube contains exactly one red subcube. In how many ways can this be done?

Reinhardt Messerschmidt

Introduction

We will show that if $n \in \{3,4,5\}$ then the number of such colourings of an $n\times n\times n$ cube, allowing for rotational symmetries, is

$$\frac{1}{24}\left((n!)^2 + 6(n!) + 8\left(1 + 2\binom{n}{3}\right)\right) = \begin{cases} 4 & \text{if } n = 3 \\ 33 & \text{if } n = 4 \\ 637 & \text{if } n = 5. \end{cases}$$

Preliminaries

The cube can be represented with the set

$$\mathcal{A} = \{1,\ldots,n\} \times \{1,\ldots,n\} \times \{1,\ldots,n\}.$$

A *colouring* is a subset C of $\mathcal{A}$ consisting of n points

$$(x_1,\ y_1,\ z_1), \quad \ldots, \quad (x_n,\ y_n,\ z_n)$$

such that $x_1,\ldots,x_n$ are distinct, $y_1,\ldots,y_n$ are distinct, and $z_1,\ldots,z_n$ are distinct. Let $\mathcal{C}$ be the set of all colourings.

Let G be the group of rotational symmetries of the cube. This group has 24 elements, which can be categorized as follows:

- *The identity rotation*: There is 1 element in this category.
- *Face rotations*: The rotations around an axis through the midpoints of two opposite faces. There are three such axes and three magnitudes of rotation (90°, 180° and 270°); therefore this category has 9 elements.
- *Edge rotations*: The rotations around an axis through the midpoints of two opposite edges. There are six such axes and one magnitude of rotation (180°); therefore this category has 6 elements.
- *Vertex rotations*: The rotations around an axis through two opposite vertices (in other words, around a diagonal of the cube). There are four such axes and two magnitudes of rotation (120° and 240°); therefore this category has 8 elements.

Let $\sim$ be the relation on $\mathcal{C}$ defined by

$$C_1 \sim C_2 \iff \sigma(C_1) = C_2 \text{ for some } \sigma \in G.$$

This is an equivalence relation. Its equivalence classes are called *orbits*.

Outline

We want to find the number of orbits of $\sim$. By *Burnside's lemma*, this is equal to

$$\frac{1}{|G|} \sum_{\sigma \in G} \bigl| \operatorname{Fix}(\sigma) \bigr|,$$

where

$$\operatorname{Fix}(\sigma) \;=\; \bigl\{ C \in \mathcal{C} : \sigma(C) = C \bigr\}.$$

We know that $|G| = 24$; therefore it remains for us to find $|\operatorname{Fix}(\sigma)|$ for each $\sigma \in G$.

The identity rotation

Suppose σ is the identity rotation. Clearly, $\operatorname{Fix}(\sigma) = \mathcal{C}$. The following procedure generates all the elements of $\mathcal{C}$:

- Choose
  $$y_1 \in \{1, \ldots, n\}, \qquad z_1 \in \{1, \ldots, n\}.$$
  This can be done in n^2 ways.
- Choose
  $$y_2 \in \{1, \ldots, n\} - \{y_1\}, \qquad z_2 \in \{1, \ldots, n\} - \{z_1\}.$$
  This can be done in $(n-1)^2$ ways.
- $\ldots$
- Choose
  $$y_n \in \{1, \ldots, n\} - \{y_1, \ldots, y_{n-1}\}, \qquad z_n \in \{1, \ldots, n\} - \{z_1, \ldots, z_{n-1}\}.$$
  This can be done in 1 way.
- Form the colouring
  $$(1,\ y_1,\ z_1), \quad (2,\ y_2,\ z_2), \quad \ldots, \quad (n,\ y_n,\ z_n).$$

It follows that

$$\bigl| \operatorname{Fix}(\sigma) \bigr| \;=\; |\mathcal{C}| \;=\; n^2 (n-1)^2 \cdots 1 \;=\; (n!)^2.$$

Face rotations

Suppose σ is the 90° clockwise rotation around the axis through the points

$$\left(1,\ \frac{n+1}{2},\ \frac{n+1}{2}\right),\quad \left(n,\ \frac{n+1}{2},\ \frac{n+1}{2}\right),$$

as viewed from the first point towards the second point. In other words,

$$\sigma\big((x,\ y,\ z)\big) \ =\ (x,\ n-z+1,\ y).$$

If C is the colouring

$$(1,\ y_1,\ z_1),\quad \ldots,\quad (n,\ y_n,\ z_n),$$

then $\sigma(C)$ is

$$(1,\ n-z_1+1,\ y_1),\quad \ldots,\quad (n,\ n-z_n+1,\ y_n);$$

therefore

$$\begin{aligned}\sigma(C)=C \quad &\Longleftrightarrow\quad y_r=n-z_r+1 \text{ and } y_r=z_r \text{ for every } r\in\{1,\ldots,n\}\\ &\Longleftrightarrow\quad y_r=z_r=(n+1)/2 \text{ for every } r\in\{1,\ldots,n\}.\end{aligned}$$

No valid colouring satisfies this condition, because $y_1,\ldots,y_n$ have to be distinct. It follows that

$$\big|\operatorname{Fix}(\sigma)\big| \ =\ 0.$$

The same holds for all the other face rotations.

Edge rotations

Suppose σ is the 180° rotation around the axis through the points

$$\left(\frac{n+1}{2},\ 1,\ 1\right),\quad \left(\frac{n+1}{2},\ n,\ n\right).$$

In other words,

$$\sigma\big((x,\ y,\ z)\big) \ =\ (n-x+1,\ z,\ y).$$

If C is the colouring

$$(1,\ y_1,\ z_1),\quad \ldots,\quad (n,\ y_n,\ z_n),$$

then $\sigma(C)$ is

$$(1,\ z_n,\ y_n),\quad \ldots,\quad (n,\ z_1,\ y_1);$$

therefore

$$\sigma(C) = C \iff z_r = y_{n-r+1} \text{ for every } r \in \{1, \ldots, n\};$$

therefore $\mathrm{Fix}(\sigma)$ consists of all colourings of the form

$$(1,\ y_1,\ y_n), \quad (2,\ y_2,\ y_{n-1}), \quad \ldots, \quad (n,\ y_n,\ y_1),$$

where $y_1, \ldots, y_n$ are distinct elements of $\{1, \ldots, n\}$. It follows that

$$\bigl|\,\mathrm{Fix}(\sigma)\bigr| \ = \ n!\,.$$

The same holds for all the other edge rotations.

Vertex rotations

Suppose σ is the 120° clockwise rotation around the axis through the points $(1, 1, 1)$ and (n, n, n), as viewed from $(1, 1, 1)$ towards (n, n, n). In other words,

$$\sigma\bigl((x,\ y,\ z)\bigr) \ = \ (z,\ x,\ y).$$

One element of $\mathrm{Fix}(\sigma)$ is

$$(1,\ 1,\ 1), \quad (2,\ 2,\ 2), \quad \ldots, \quad (n,\ n,\ n).$$

The following procedure generates all the other elements of $\mathrm{Fix}(\sigma)$, remembering that $n \in \{3, 4, 5\}$:

- Choose distinct $r, s, t \in \{1, \ldots, n\}$. The number of ways that this can be done is
$$\binom{n}{3}.$$
- Form the two colourings
$$(r,\ s,\ t), \quad (t,\ r,\ s), \quad (s,\ t,\ r), \quad (u,\ u,\ u), \quad u \in \{1, \ldots, n\} - \{r, s, t\}$$
and
$$(r,\ t,\ s), \quad (s,\ r,\ t), \quad (t,\ s,\ r), \quad (u,\ u,\ u), \quad u \in \{1, \ldots, n\} - \{r, s, t\}.$$

It follows that

$$\bigl|\,\mathrm{Fix}(\sigma)\bigr| \ = \ 1 + 2\binom{n}{3}.$$

The same holds for all the other vertex rotations.

The relativistic mass formula near light speed

Mako Sawin

Introduction

In the pursuit of understanding the profound implications of Einstein's theory of special relativity, one encounters the intriguing concept of relativistic mass increase as objects approach the speed of light. The formula governing this phenomenon, $m = m_0/\sqrt{1 - v^2/c^2}$, encapsulates the essence of how mass dynamically changes with velocity. However, this article sets out to investigate the behaviour of the relativistic mass formula specifically when objects accelerate close to, or even potentially exceed, the speed of light.

Through mathematical analysis, we aim to shed light on the implications and insights derived from this exploration, offering a deeper understanding of the interplay between mass and velocity in extreme regimes.

The formula for relativistic mass increase is given by

$$m = \frac{m_0}{\sqrt{1 - \left(\frac{v}{c}\right)^2}}, \tag{1}$$

where

- m is the relativistic mass,
- m_0 is the rest mass (mass of the object when it's at rest),
- v is the velocity of the object,
- c is the speed of light in a vacuum (exactly $299,792,458$ metres per second by definition of the metre).

As the velocity v approaches the speed of light c, the denominator approaches zero, causing the relativistic mass m to increase towards infinity.

Part 1

We presume that we have a mass m_0 of 1 kilogram, and its velocity is approaching the speed of light, $v = 2.990 \times 10^8$ metres per second. For simplicity, let's consider the speed of light c as 3.000×10^8 metres per second. Therefore

$$m = \frac{1}{\sqrt{1 - \left(\frac{2.990 \times 10^8}{3.000 \times 10^8}\right)^2}} \approx 12.26\,\text{kg}$$

Part 2

Let's presume that the mass exceeds the speed of light by 0.010×10^8 metres per second, $v = 3.010 \times 10^8$. Then

$$m \;=\; \frac{1}{\sqrt{1-\left(\dfrac{3.010\times10^8}{3.000\times10^8}\right)^2}} \;=\; \frac{1}{\sqrt{-0.006678}} \;\approx\; 12.24\,i\text{ kg}.$$

In examining the relativistic mass formula under varied velocity conditions, our investigation reveals intriguing outcomes. When the velocity approaches the speed of light by a marginal increment of 10^6 metres per second, the resulting relativistic mass reaches approximately 12.26 kg, showcasing the significant impact of relativistic effects even at modest velocities. However, when the velocity exceeds the speed of light by 10^6 metres per second, the formula yields an imaginary value $\approx 12.24\,i$ kg, hinting at the theoretical constraints imposed by the laws of physics.

These contrasting outcomes underscore the delicate balance between theory and practical application in relativistic scenarios. While the formula elucidates the transformative effects of velocity on mass, it also underscores the theoretical boundaries inherent in relativistic physics. Our exploration underscores the necessity for nuanced interpretations and further investigation to unravel the complexities of mass-velocity relationships in the relativistic regime.

Problem 319.2 – Permutations

Tony Forbes

Let n be a positive integer and let P be a permutation of $\{1, 2, \ldots, n^2\}$ that contains no increasing or decreasing subsequence of length $n+1$. Show that the first and last elements of P must be at least n.

Problem 319.3 – Sum

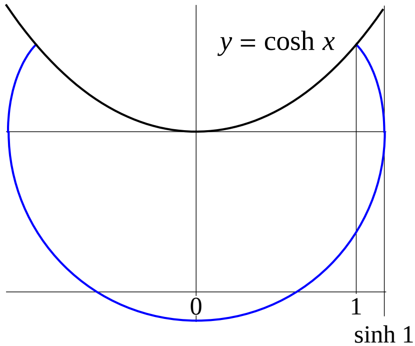
Show that

$$\frac{1}{1\cdot3\cdot5}+\frac{1}{7\cdot9\cdot11}+\frac{1}{13\cdot15\cdot17}+\ldots \;=\; \frac{\log 3}{16}.$$

The factors in the denominators run through the odd positive integers.

Solution 316.4 – Goat

There is a field bounded by the function $\cosh x$ with its y-axis pointing north. A goat is tethered to $(0, 1)$ by a rope of length $\sinh 1$ and the animal is otherwise free to graze south of the curve. What is the area of the part of the field that the goat can access?



Tony Forbes

Recall that I claimed that this problem admits an exact answer involving only elementary functions. For the area, we have

$$A = A_1 + 2A_2,$$

where A_1 is a semicircle of radius $\sinh 1$, and A_2 is the area bounded by $\cosh x$, the blue curve from $(1, \cosh 1)$ to $(\sinh 1, 1)$ and the line from $(0, 1)$ to $(\sinh 1, 1)$. Clearly,

$$A_1 = \frac{\pi(\sinh 1)^2}{2} = 2.16942,$$

and it remains to compute the non-trivial part, A_2.

Let's do some high-school calculus. Suppose $u \in (0, 1)$ and consider what happens when we advance from u to $u + du$, corresponding to the goat moving from G to H along the blue curve. There is a diagram on the next page.

Ignoring terms involving $(du)^2$ and higher powers, we see that the tether sweeps out an area that is approximately a right-angled triangle with sides $\sinh 1 - \sinh u$ and $|GH|$. To compute $|GH|$, we have

$$\begin{aligned} a(u)^2 + b(u)^2 &= (\sinh 1 - \sinh u)^2, \\ b(u) &= a(u)\cosh'(u) = a(u)\sinh u, \end{aligned}$$

which can be solved to get

$$a(u) = \frac{\sinh 1 - \sinh u}{\cosh u}.$$

Moreover, noting that $a'(u) \le -1$ for $u \in [0,1]$,

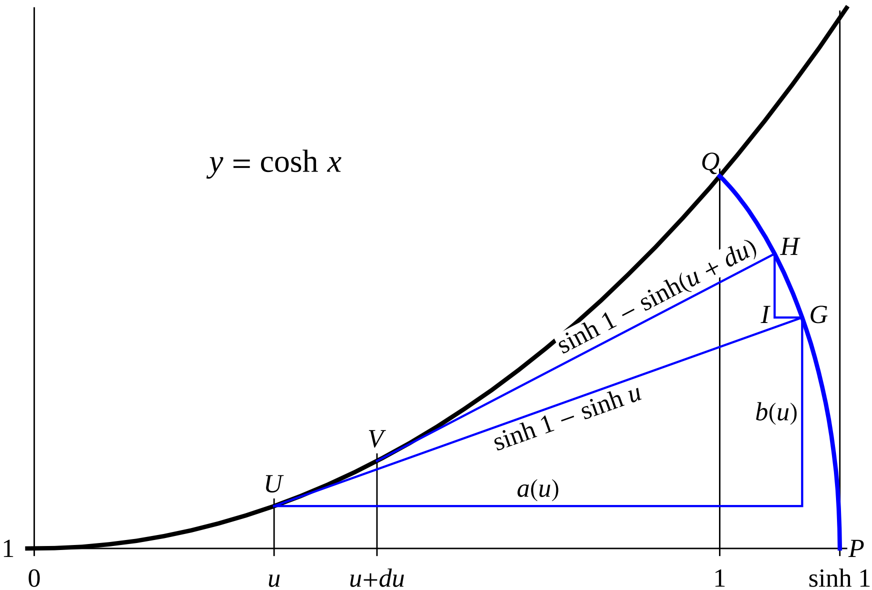
$$\begin{aligned}
|GI| &= u + a(u) - (u + du + a(u+du)) = -(a'(u)+1)\,du,\\
|IH| &= \frac{|GI|}{\sinh u} = -\frac{a'(u)+1}{\sinh u}\,du,\\
|GH| &= \sqrt{|GI|^2 + |IH|^2} = -\frac{a'(u)+1}{\tanh u}\,du,
\end{aligned}$$

and the area of 'triangle' $UGHV$ is approximately

$$\begin{aligned}
dA_2 &= \frac{1}{2}\,|GH|(\sinh 1 - \sinh u) = \frac{(a'(u)+1)(\sinh u - \sinh 1)}{2\tanh u}\,du\\
&= \frac{(e^2 - 2e\,\sinh(u) - 1)^2}{8e^2\cosh u}\,du,
\end{aligned}$$

which can be integrated to get

$$\begin{aligned}
A_2 &= \int_0^1 \frac{(e^2 - 2e\,\sinh(u) - 1)^2}{8e^2\cosh u}du\\
&= \frac{1}{4e^2}\left((e^4 - 6e^2 + 1)\arctan\left(\frac{e-1}{e+1}\right) + (e^3 - e)\log\left(\frac{4e^3}{(1+e^2)^2}\right)\right)\\
&= 0.242792.
\end{aligned}$$



Eigenvalues of certain matrices

Tony Forbes

Here is something interesting the motivation for which was a short article by Nick Trefethen: Discrete and Continuous, *LMS Newsletter* **510** (Feb 2024).

Let n be a positive integer. For $k = 1, 2, \ldots, 2n$, construct a $(2n+1) \times (2n+1)$ matrix $M(n, k)$ as follows.

> The elements on the main diagonal are n, $n-1$, …, 0, 1, …, n.
>
> The elements on the k diagonals immediately above and immediately below the main diagonal are all 1.
>
> All other elements are zero.

For example, this what $M(7, 4)$ looks like:

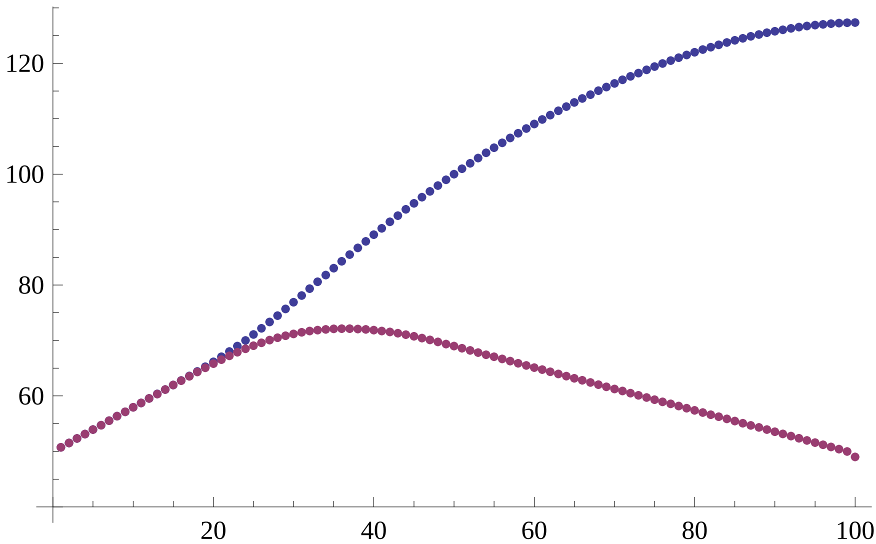
$$\begin{bmatrix}
7 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
1 & 6 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
1 & 1 & 5 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
1 & 1 & 1 & 4 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
1 & 1 & 1 & 1 & 3 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 1 & 1 & 1 & 1 & 2 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 2 & 1 & 1 & 1 & 1 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 3 & 1 & 1 & 1 & 1\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 4 & 1 & 1 & 1\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 5 & 1 & 1\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 6 & 1\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 7
\end{bmatrix}.$$

We are interested in the two largest eigenvalues. Unless $k = 0$ or the matrix is small, computing eigenvalues can be a tricky process, which is best left to MATHEMATICA. When $n = 7$ the calculations yield the following.

$k = 1$	7.74619	7.74619	$k = 2$	8.53687	8.53258
$k = 3$	9.41968	9.25581	$k = 4$	10.5384	9.72839
$k = 5$	11.7651	9.86893	$k = 6$	12.938	9.72687
$k = 7$	14	9.37665	$k = 8$	14.9373	8.98803
$k = 9$	15.7537	8.59848	$k = 10$	16.4557	8.20740
$k = 11$	17.0440	7.81361	$k = 12$	17.5142	7.41421
$k = 13$	17.8550	7	$k = 14$	18.0453	6

What stands out quite prominently is that the two largest eigenvalues of $M(7,1)$ are very nearly equal. Thereafter, the largest eigenvalue of $M(7,k)$ increases as k increases. However, the other one behaves differently. The second eigenvalue increases whilst trying to maintain near equality with the first. It reaches a maximum at $k = 5$, then decreases to 7 at $k = 13$ and 6 at $k = 14$.

The effect is more striking when shown graphically for a large n, as illustrated below for $n = 50$.



The two largest eigenvalues of $M(50,k)$ for $k = 1, 2, \ldots, 100$.

We invite the reader to prove three theorems suggested by the entries for $k = 7$, 13 and 14 in the table on the previous page.

(i) The largest eigenvalue of $M(n,n)$ is $2n$.

(ii) The second-largest eigenvalue of $M(n, 2n-1)$ is n.

(iii) The second-largest eigenvalue of $M(n, 2n)$ is $n - 1$.

Problem 319.4 – Points in a disc

There are 7 points in the unit disc $\{(x,y) : x^2 + y^2 \leq 1\}$. Show that either one of the points is the centre of the disc, or there are two points that separated by distance less than 1.

Getting our ducks in a row

Danny Roach

"Where's the Ducks?"

This was the question with which I hit Jenny, our patient OU tutor, upon my return from lunch on the afternoon of the third day of the M500 Society's annual Maths revision weekend.

"Ah, so you've been speaking to last year's students then?" she queried with a smile.

What's this got to do with a mathematics weekend? Well stick with me and I'll tell you more on the ducks later . . .

Every year, the M500 Society runs a revision weekend, covering the majority of the courses run that year. Anyone can attend, and with my exam looming I had registered to attend the sessions covering MST124 – Essential Mathematics 1. Having little formal mathematical background, I figured I needed all the help I could get to pass the exam well and so this for me was something I could not afford to miss.

When I arrived on Friday evening I was warmly welcomed but was surprised at the small class size. There were a diverse mix of backgrounds amongst the 8 people on the MST124 revision stream, although I'd estimate there were a few hundred delegates in total studying various different maths courses. Given the number of students studying MST124 (it's a compulsory module in many OU Maths and Physics degrees) I was surprised at the low take up and wondered, do people actually know this weekend exists? That's in part why I wanted to write this piece, as I and my classmates, found this weekend invaluable.

Jenny Oldroyd was our tutor and from the moment we set foot in the classroom on Friday evening she was warm, entertaining, patient, and clear. She explained things that I've vaguely learned from the module material this year, but now understand much more thoroughly due to the nature of a face-to-face tutorial session – OU take note, we could do with some face-to-face tutorial options, even if it's only 4 or 5 sessions per year. I can't emphasis enough what a huge difference it has made, really solidifying my learning in some of my personal nemesis topics – Integration by Parts, Binomial Expansion, that's you I'm talking about!

There were also plenty of revision and exam tactics discussed, and some helpful calculator hints, for example, if you have two calculators have one set up in radians and one in degrees (and label them) to save time and reduce errors during the exam. I've discovered my calculator can do so much more than arithmetic thanks to this weekend!

The classroom dynamic was fun with plenty of banter between students and teacher, as we students competed with each other for the coveted title of 'Who can ask the most inane question?' By the end of the third day we had built a natural bond and all swapped email addresses. I'm sure some of us will stay in touch as we continue our OU journeys.

The organisation, led by Judith from the M500 Society, was superb, as was the venue. The Kents Hill Conference Centre which is adjacent to the OU MK campus was a great choice, with clean bright rooms, good food and drink, and nice open spaces in which to take advantage of this years first real sight of the Sun (for this Northern boy). Speaking of Northern, several students also saw the Northern Lights on the Friday evening and shared some stunning photos on social media.

The cost was very reasonable too. For two nights full board, including tea/coffee/snacks during the day, breakfast, dinner and tea (southerners note, that's the correct name for the three meals), it was £295 – this included around 15 hours of mathematics tuition, which I think is outstanding value. There were other options for those who didn't wish to stay over which reduced the cost further. So, what about those ducks?

Well Jenny gave out small rubber ducks last year to her cohort, explaining that it helps to say maths out loud as you're doing it and reduces the chance of a mistake. Not everyone has someone there to be able to talk to, and so she advocates, 'say it to the duck.' As a former programmer who always verbalises difficult problems, I can confirm that this is a great method to help eliminate small mistakes like getting minus's wrong.

Chanting "Three minus minus two is five." Really helps avoid those silly errors, although I characteristically managed to throw some in this weekend, just to keep Jenny on her toes!

Although Jenny had forgotten to bring her ducks with her this year, I already have one of my own which was bought for me by my eldest daughter. I will now keep him on my desk for future maths conversation and validation.

My thanks to everyone at the M500 Society for a superbly run weekend. I've learned maths I didn't know, solidified things I thought I knew and done it all with a smile and a laugh with some new friends. I can't recommend this weekend enough, everyone who I met had the same silly grin on their face, and went away with a new found confidence in their maths. And that is about as good an outcome as you can get.

Dan Roach is an IT Consultant, Flight Instructor and Open University physics undergraduate based in the North West, UK. He is the author of *Inside the Cyclone* and blogs about aviation, space and science at danroachauthor.co.uk. You can follow him on X, @dannydenfisch or on Facebook, @danroachauthor,

`http://danroachauthor.co.uk`,

`http://facebook.com/danroachauthor`.

Available now: *Inside the Cyclone* on Amazon in paperback and Kindle e-book.

Letter

M500 217

Hi Tony,

Looking at the integration by parts on page 18 of M500 **317** reminds me of similar treatment I did and submitted previously. My thoughts at the time was the format for a computer project that I was working on. For comparison of this mathematical treatment, readers may wish to look at these previous issues:

249, page 12: The CDF of a Standard Normal Distribution.

256, page 1: The Chi-Squared Distribution.

263, page 10: Integration of Polynomial/Exponential Functions.

The relevant issues can be downloaded from the M500 website.

Secondly, looking at Problem 317.7, Nine cards: the first nine numbers add up to an odd number; so a mathematical solution is impossible, and therefore the solution can only be achieved by trickery. So far I have found three ways of doing this. Invert the 6, 1 covers the 5 or 2 covers the 7.

Regards,

Ken Greatrix

Solution 317.5 – Approximation

Show that for small x,

$$\exp(\tan x) \;=\; \sqrt{\frac{1+x}{1-x}} + O(x^5).$$

Thus, for example, $e^{2\tan 0.001} \approx 1.002002002002002$.

David Sixsmith

Using standard Taylor series, we have

$$\begin{aligned}
\log\left(\sqrt{\frac{1+x}{1-x}}\right) &= \frac{1}{2}(\log(1+x) - \log(1-x)) \\
&= \frac{1}{2}\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + O(x^5) + x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + O(x^5)\right) \\
&= x + \frac{x^3}{3} + O(x^5) \\
&= \tan(x) + O(x^5).
\end{aligned}$$

Taking exponentials and, using the Taylor series for exp, we get

$$\begin{aligned}
\exp(\tan(x)) &= \exp\left(\log\left(\sqrt{\frac{1+x}{1-x}}\right) + O(x^5)\right) \\
&= \sqrt{\frac{1+x}{1-x}}.\,(1 + O(x^5)) \\
&= \sqrt{\frac{1+x}{1-x}} + O(x^5),
\end{aligned}$$

as required.

But this seems like cheating somehow, and not in the spirit of a nice question. Can a solution be found that doesn't use differentiation or Taylor series?

Problem 319.5 – Factorial factorial

How many zeros are at the end of $(n!)!$?

If that's difficult, do $n!$ first.

If that's difficult, try a special case, say 100!.

04.00

Problem 319.6 – Random triangles

Tony Forbes

There are n points chosen at random in a rectangle with sides $a \geq 1$ and 1. Of the $n(n-1)(n-2)/6$ triangles that can be made from the points taken three at a time, what is the expected number that have all three angles less than 90 degrees.

Front cover 100 triangles in a 25×20 rectangle, some red, some green.