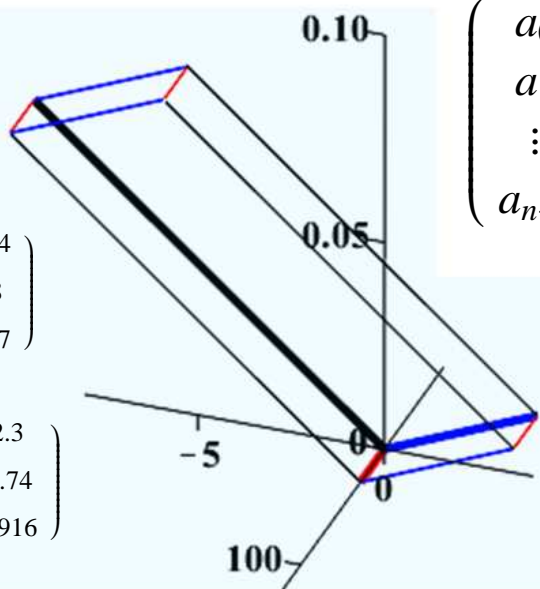
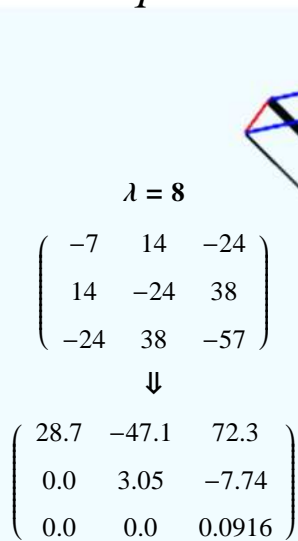




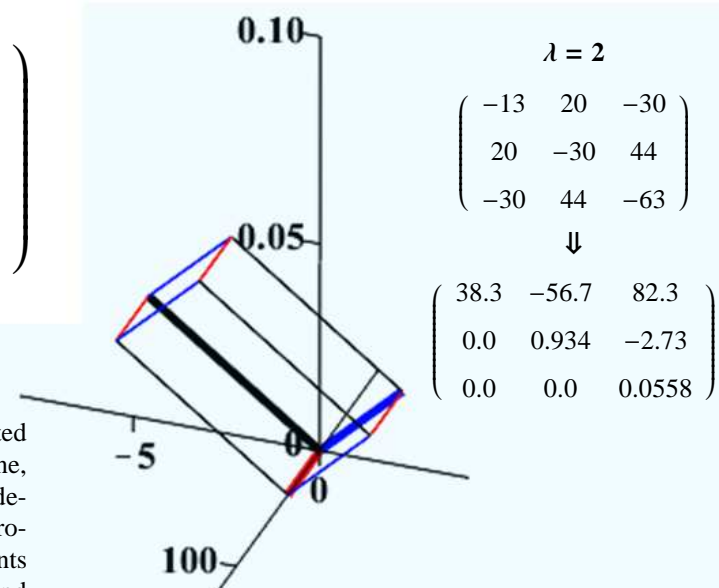
# THEOREM OF THE DAY

**Sylvester's Catalecticant** Let  $f(x, y) = \sum_{i=0}^n a_i \binom{n}{i} x^{n-i} y^i$  and  $h(x, y) = \sum_{j=0}^r c_j x^{r-j} y^j$ , with  $a_i, c_j \in \mathbb{C}$ , be binary forms in two variables  $x$  and  $y$ , of degree  $n$  and  $r$ , respectively, and suppose that  $h(x, y)$  may be written as a product of  $r$  pairwise distinct linear factors:  $h(x, y) = \prod_{k=1}^r (-\beta_k x + \alpha_k y)$ . Then a necessary and sufficient condition for  $f(x, y)$  to be written as a weighted sum of  $n$ -th powers of the linear factors  $(\alpha_k x + \beta_k y)$ , i.e.  $f(x, y) = \sum_{k=1}^r \lambda_k (\alpha_k x + \beta_k y)^n$ ,  $\lambda_k \in \mathbb{C}$ , is that the coefficients of  $f$  and  $h$  satisfy the matrix equation

$$\begin{pmatrix} a_0 & a_1 & \cdots & a_r \\ a_1 & a_2 & \cdots & a_{r+1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n-r} & a_{n-r+1} & \cdots & a_n \end{pmatrix} \cdot \begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_r \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$



Catalecticant matrices for  $f(x, y) = \lambda(x-y)^4 + (x-2y)^4 - (2x-3y)^4$ , for  $\lambda = 8, 2$ , treated as triples of column vectors, rotated so that the first two lie on the  $x$ -axis and in the  $xy$ -plane, respectively. And the parallelepipeds they induce: as  $\lambda$  decreases to zero the parallelepipeds shrink and 'flatten' progressively such that volumes, identical to the determinants of the matrices, also decreases to zero; at which point, and not otherwise, Sylvester's matrix equation is satisfied.



The above illustration gives a geometric interpretation of minimal quartic representation: can the binary quartic form  $f(x, y) = \lambda(ax + by)^4 + \mu(cx + dy)^4 + \nu(ex + fy)^4$ , a sum of three fourth powers of linear factors, be represented as a sum of just two? We can calculate the *catalecticant* of  $f(x, y)$ , i.e. the determinant of its catalecticant matrix, as  $\lambda\mu\nu(ad - bc)^2 (af - be)^2 (cf - de)^2$ . This can only vanish in degenerate cases: one of the linear factors is absent or is a multiple of another, i.e., when and only when the sum is of two fourth powers.

So catalecticant matrix rank gives a test for a reduced representation of  $f$  as a sum of powers. But the introduction of the form  $h$  gives more: an algorithm for the reduction. Consider  $f(x, y) = x^3 - 6xy^2$ . Sylvester's equation is  $\begin{pmatrix} 1 & 0 & -2 \\ 0 & -2 & 0 \end{pmatrix} \cdot (c_1 \ c_2 \ c_3)^T = (0 \ 0 \ 0)^T$ , which solves to give  $c_1 = 2c_3 = 2$ , say, and  $c_2 = 0$ . Now we can construct  $h(x, y) = 2x^2 + 0xy + y^2$ ; and factorise it as  $h(x, y) = (\sqrt{2}ix + y)(-\sqrt{2}ix + y)$ . So for some  $\lambda_1, \lambda_2$  we can write  $f(x, y) = \lambda_1(x - \sqrt{2}iy)^3 + \lambda_2(x + \sqrt{2}iy)^3$  and indeed  $\lambda_1 = \lambda_2 = 1/2$  gives the desired expansion.

The representability of binary forms as sums of powers was largely resolved by Sylvester who introduced the catalecticant for this purpose in 1851. Minimal representability of forms in general belongs to *Waring's problem for forms* and continues to be a topic of deep research.