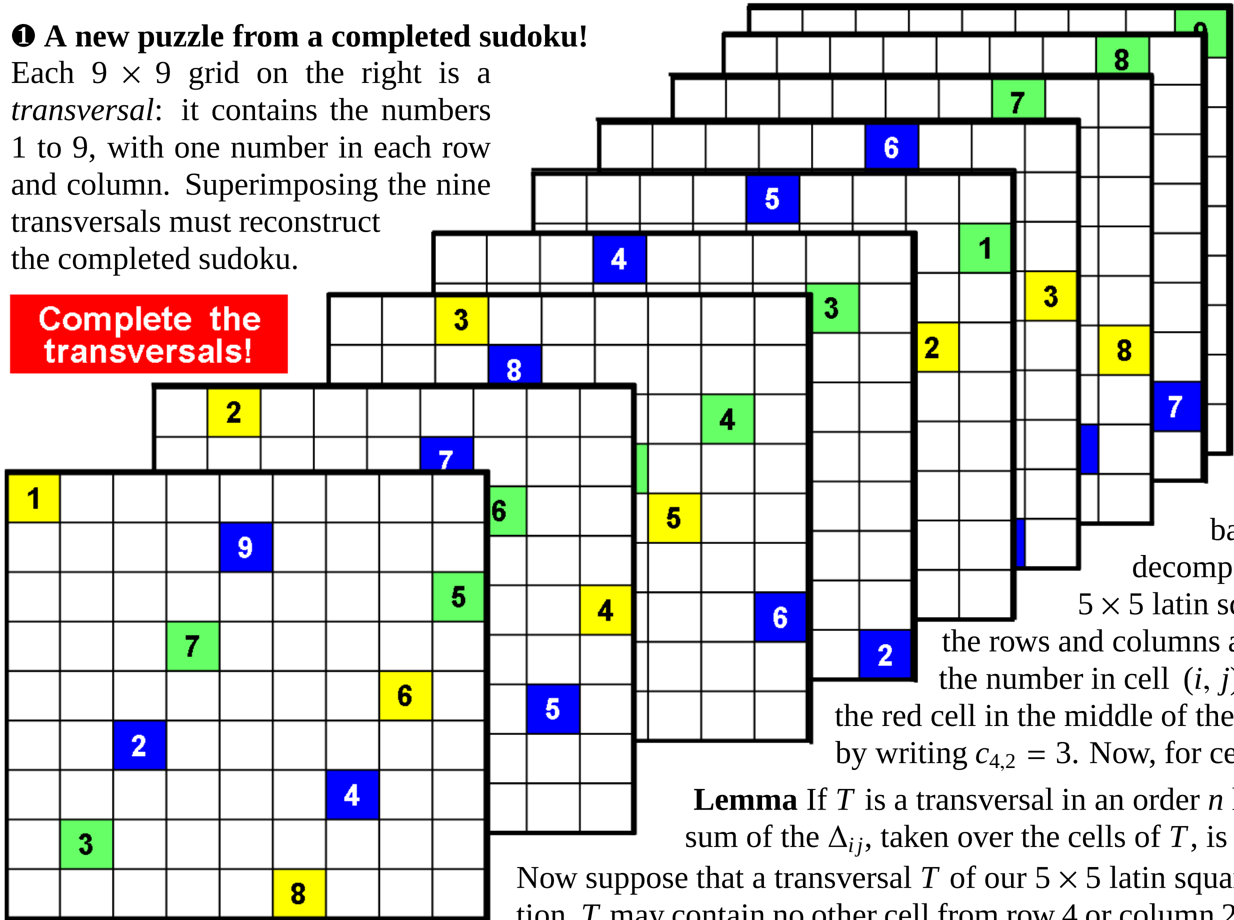
THEOREM OF THE DAY

The Existence Theorem for Bachelor Latin Squares *Bachelor latin squares exist for all orders except for 1 and 3.*

1	2	3	4	5	6	7	8	9
5	6	4	8	9	7	2	3	1
9	7	8	3	1	2	6	4	5
4	5	6	7	8	9	1	2	3
8	9	7	2	3	1	5	6	4
3	1	2	6	4	5	9	7	8
7	8	9	1	2	3	4	5	6
2	3	1	5	6	4	8	9	7
6	4	5	9	7	8	3	1	2

❶ A new puzzle from a completed sudoku! Each 9×9 grid on the right is a *transversal*: it contains the numbers 1 to 9, with one number in each row and column. Superimposing the nine transversals must reconstruct the completed sudoku.



❷ Orthogonal Mates A latin square of order n contains n symbols repeated n times so as to appear once in each row and column of an $n \times n$ grid. Completed sudoku puzzles are a special type of latin square of order 9. Some latin squares may be 'decomposed' into transversals, as in our puzzle here. In this case alone, an *orthogonal mate* may be constructed by replacing all symbols in the i-th transversal with the i-th symbol. Latin squares with no orthogonal mates are called *bachelors*!

	0	1	2	3	4
0	1	0	2	3	4
1	0	2	1	4	3
2	2	3	4	0	1
3	3	4	0	1	2
4	4	1	3	2	0

❸ Confirmed Bachelors A latin square may have a cell which can lie in no transversal. Then it may be called a *confirmed* bachelor since it can certainly never decompose into transversals! In the above 5×5 latin square the symbols are $0, \ldots, 4$ and the rows and columns are indexed similarly. Write c_{ij} for the number in cell (i, j), indexed by row i and column j; the red cell in the middle of the last row, for example, is specified by writing $c_{4,2} = 3$. Now, for cell (i, j), write $\Delta_{ij} = c_{ij} - i - j$.

Lemma If T is a transversal in an order n latin square, and n is odd, then the sum of the Δ_{ij}, taken over the cells of T, is a multiple of n.

Now suppose that a transversal T of our 5×5 latin square contains the red cell. By definition, T may contain no other cell from row 4 or column 2; nor the other cell having value 3; nor two from cells $(0, 0)$, $(0, 1)$ and $(1, 0)$. But $\Delta_{4,2} = 3 - 4 - 2 = -3$, and all unshaded cells have $\Delta = 0$, so we see that it is impossible for T to satisfy the Lemma. Conclusion: our latin square is a confirmed bachelor!

The question of whether bachelor squares exist for all orders (except $n = 1$ and $n = 3$, which do not admit bachelors) goes back to Leonhard Euler who observed in 1779 that the addition table modulo n has no transversals when n is even. The case of odd-order latin squares is less tractable: a conjecture attributed to HJ Ryser says that these squares always have at least one transversal. However, Henry B. Mann showed in 1944 that bachelors exist for all $n \equiv 1 \pmod 4$. The case $n \equiv 3 \pmod 4$ was resolved more than 60 years later, in 2006, by Anthony B Evans and, independently, by Ian Wanless and Bridget Webb, who used the simple lemma, above right, to show that *confirmed* bachelors exist for all $n \neq 1, 3$.

Web link: arxiv.org/abs/0903.5142. The sudoku transversals puzzle is solved here.

Further reading: *Latin Squares: New Developments in the Theory and Applications* by J. Dénes and A.D. Keedwell, North-Holland, 1991.