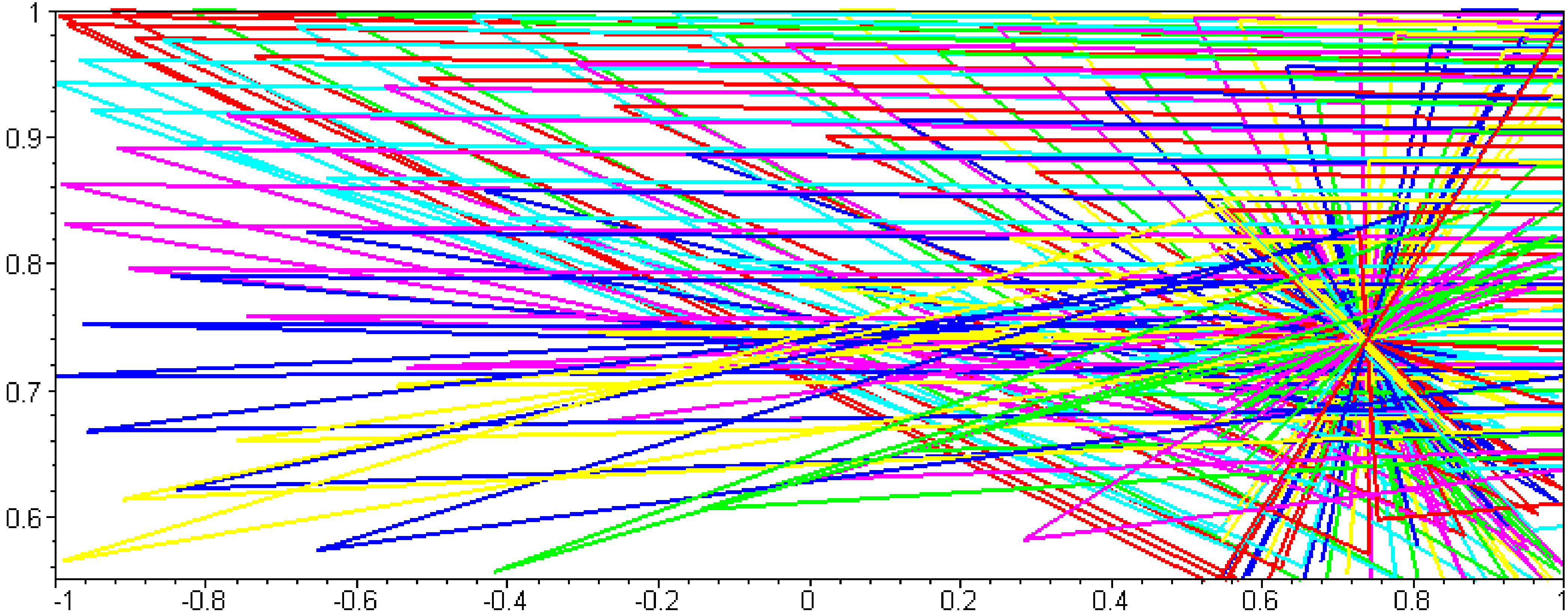
THEOREM OF THE DAY

The Lindemann-Weierstrass Theorem *If $\alpha_1, \ldots, \alpha_n$, $n \geq 1$, are algebraic numbers which are linearly independent over $\mathbb{Q}$, then $e^{\alpha_1}, \ldots, e^{\alpha_n}$ are algebraically independent; that is, any rational polynomial $P(z_1, \ldots, z_n)$, having algebraic coefficients, for which $P(e^{\alpha_1}, \ldots, e^{\alpha_n}) = 0$, must be identically zero.*



A simple application of this theorem tells us that when nonzero α is algebraic, $\cos(\alpha)$ is not, i.e. is transcendental. Observe, first, that αi solves the equation $x^2 + \alpha^2 = 0$, and so is algebraic. Now, suppose that $\cos(\alpha)$ is an algebraic number, say, β, and define the rational polynomial $P(z) = (z^2 - 2\beta z + 1)/2z$. Then $P(z)$ has algebraic coefficients and is obviously not identically zero, but $P(e^{i\alpha}) = 0$, via the identity $(e^{iz} + e^{-iz})/2 = \cos z$. But then Lindemann-Weierstrass contradicts the assumption that α is algebraic. We may continue and deduce, for example, that the unique real number solution D to the equation $\cos(x) = x$ is also transcendental, since if D is algebraic and solves the equation then $\cos(D) = D$ is transcendental: another contradiction! Samuel R. Kaplan (Mathematics Magazine, Feb. 2007) tells how the number D (≈ 0.7390851332) acquired the sobriquet "Dottie Number" after the wife of mathematician Paul Blanchard who drew his attention to a remarkable fact: starting from any real number x, repeated applications of the cosine function will eventually converge to D. Blanchard recognised and proved that D is a universal attractor! This is illustrated above in two dimensions by iterating the cosine for each of the pair of numbers $(x, 1 - x/100)$ for $x = 1, \ldots, 100$.

Following Charles Hermite's breakthrough 1873 proof of the transcendence of e, Ferdinand von Lindemann proved in 1882 that $e^{i\tau/2} + 1 = 0$, $\tau = 2\pi$, implied the transcendence of $\tau/2$, and conjectured the more general result stated above, proved by Karl Weierstrass in 1885.

Web link: arxiv.org/abs/2306.14352

Further reading: *Making Transcendence Transparent* by Edward B. Burger and Robert Tubbs, Springer, 2004.