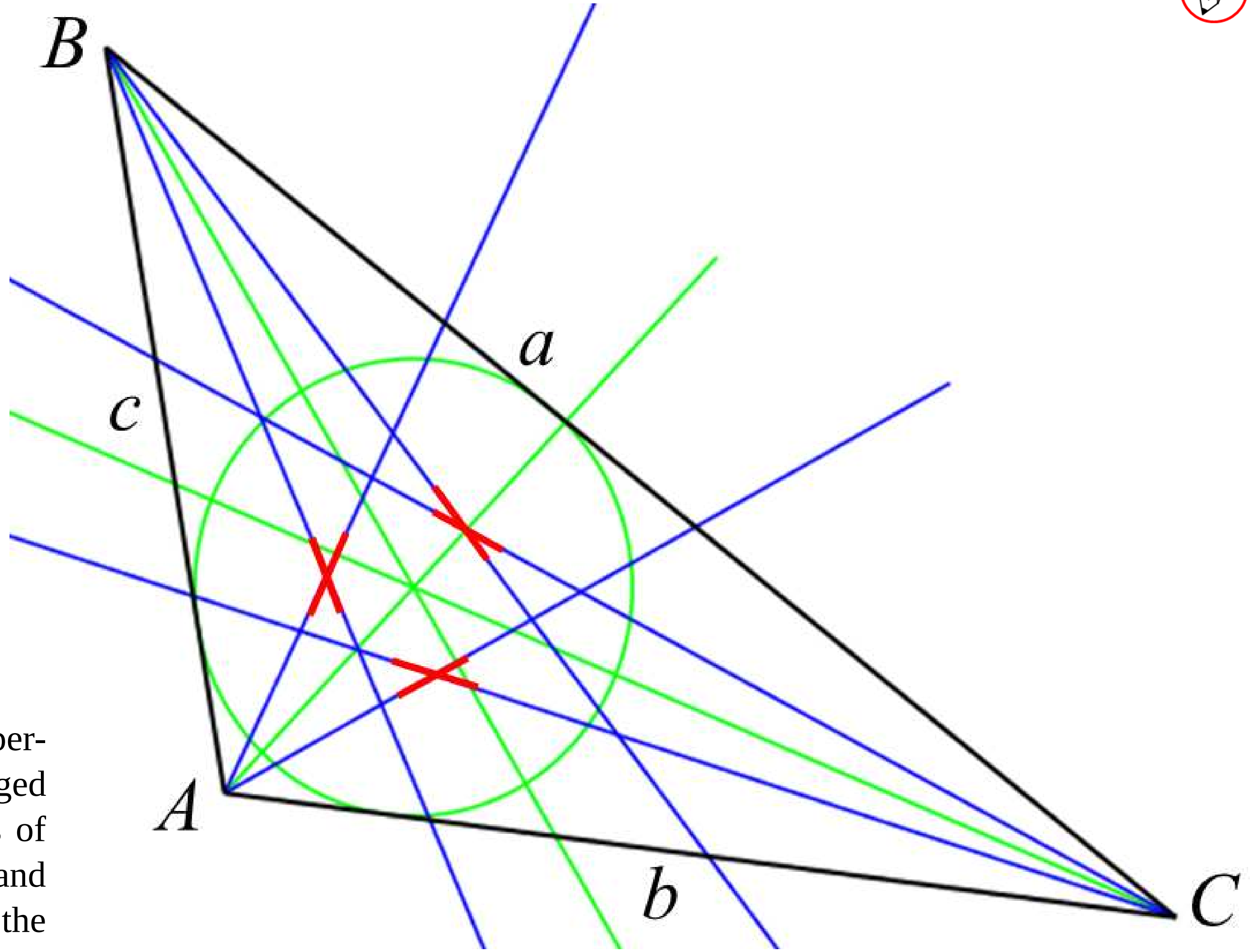
THEOREM OF THE DAY

Morley's Miracle *In any triangle the three points of intersection of adjacent trisectors of its angles form an equilateral triangle.*



The blue lines in the image on the right are the angle trisectors. The adjacent pairs intersect as highlighted and the theorem asserts that these intersections are pairwise equidistant. We can calculate this distance: intriguingly it involves the triangle's three angle bisectors, shown in green. Their unique point of intersection is the **incentre** of the triangle. It is the centre of the **incircle**, the unique circle tangent to all three triangle sides. This is also plotted in green. If its radius, the **inradius** is R, and the angles of the triangle are $\angle A$, $\angle B$ and $\angle C$ then the adjacent trisector intersections are at common distance

$$8R\sin\left(\frac{\angle A}{3}\right)\sin\left(\frac{\angle B}{3}\right)\sin\left(\frac{\angle C}{3}\right).$$

This further gives us the area of the so-called **Morley triangle** given that an equilateral triangle of side x has area $x^2\sqrt{3}/4$. (For completeness we may state the formulae for the incentre I_c and inradius I_r of a triangle whose vertices are coordinate pairs A, B and C, with opposite side lengths a, b and c: $I_c = (aA+bB+cC)/(a+b+c)$ and $I_r = 2(\text{area of } ABC)/(a+b+c)$.)

The diagram here seems quite busy! But this is perhaps appropriate since Morley's discovery emerged from his research into the incidence properties of collections of lines in the planes: the inscribed and circumscribed circles of points of intersection, the intersections of these circles, and so on.

It might seem surprising that such a striking property of triangles should post-date Euclid by over two thousand years. Or perhaps, inversely, this elapse of time might be witness to the depth of Morley's research. We may also surmise that, under the influence of Greek mathematics, the concern had long been to discover *how* to calculate angle trisectors before investigating their properties.

Web link: www.cut-the-knot.org/triangle/Morley/

Further reading: *Complex Numbers from A to ...Z,* by Titu Andreescu and Dorin Andrica, Birkhauser Boston, 2005, chapter 4, section 13.