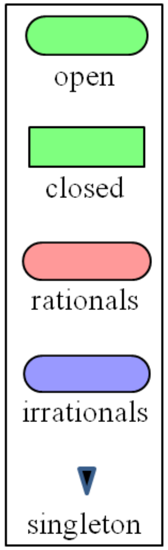
THEOREM OF THE DAY

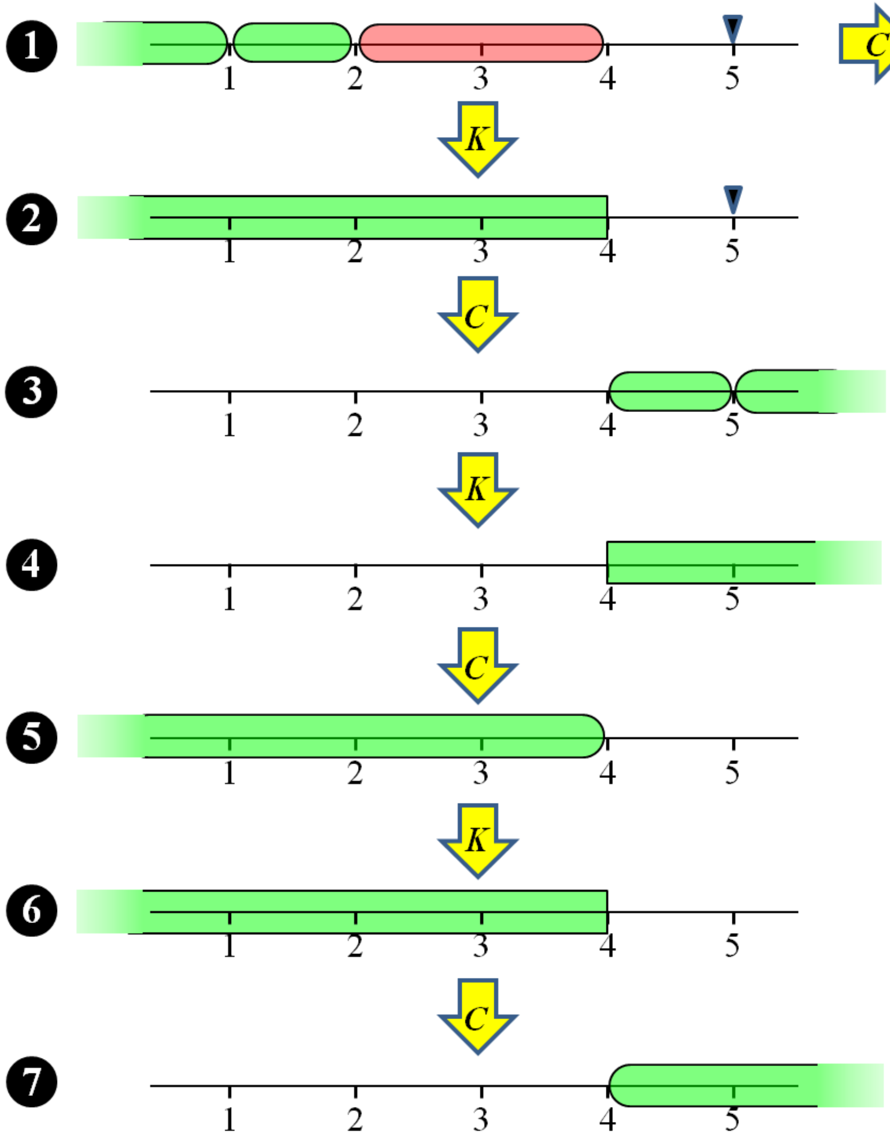
Kuratowski's 14-Set Theorem *Let $T = (S, \mathcal{T})$ be a topological space and for any subset X of S, denote by $C(X)$ the complement $S \setminus X$ of X, and by $K(X)$ the topological closure of X. Starting with an arbitrary subset of S, apply C and K repeatedly in any order; then the number of different sets that may be produced is at most 14.*

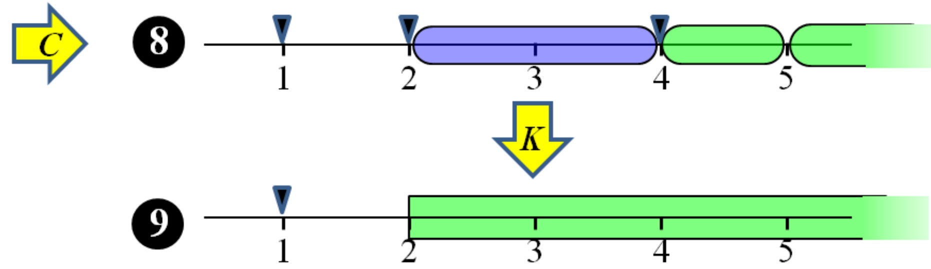
A **topological space** $T = (S, \mathcal{T})$ consists of a set S together with a collection $\mathcal{T}$ of subsets of S satisfying

1. $S, \emptyset \in \mathcal{T}$;
2. $\mathcal{T}$ is closed under taking intersections and unions.

The sets of $\mathcal{T}$ are called the **open sets** of the topology. If $X \subseteq S$ then $x \in S$ is a **limit point** of X if every open set containing x also contains some element of X other than x. Then $K(X)$ is defined to be the union of X and all limit points of X.







In our example, the topological space is the real number line, with open sets being open intervals (a, b). The set depicted top left is $(-\infty, 1) \cup (1, 2) \cup \mathbb{Q}(2, 4) \cup \{5\}$, with $\mathbb{Q}(2, 4)$ denoting the set of rational numbers in the open interval $(2, 4)$. Operators K and C are applied to produce eight new sets but this extends to the maximum possible fourteen: set 9 and a further five sets on the right are precisely the reflections of sets $2, 3, \ldots, 7$ about the point 3.

The limit of 14 expresses the fact that the **monoid** of all words on the alphabet $\{C, K\}$ has size 14. This follows because

1. $C(C(X)) = X$ (self-inverse); so any occurrence of subword CC can be removed;
2. $K(K(X)) = K(X)$ (idempotent); so any occurrence of subword KK can be replaced by K.
3. $K(C(K(C(K(C(K(x))))))) = K(C(K(X)))$; so any occurrence of subword $KCKCKCK$ can be replaced by KCK. This is not obvious. It can be seen in operation in the sequence of sets $2, 3, \ldots, 7$: applying K to set 7 will take us back to set 4.

Thus, a long word (sequence of operators) will reduce to one of length at most 7: e.g. $CKKCCCKCKKCKCCKC$ reduces, through two applications of the self-inverse rule and three of idempotency, to $CKCKCKCKC$, and thence, via rule 3, to $CKCKC$. This is the set $(2, \infty)$ (which would be set number 12 in the proposed extension of our illustration).

The truth of this result, proved by Kazimierz Kuratowski in his doctoral dissertation (1921), is a topological fact; the reason *why* it is true comes from the theory of monoids or formal language theory.

Web link: B.J. Gardner and M. Jackson at nzjm.math.auckland.ac.nz/index.php/Volume_38_2008

Further reading: *Single Digits: In Praise of Small Numbers* by Marc Chamberland, Princeton University Press, 2015.