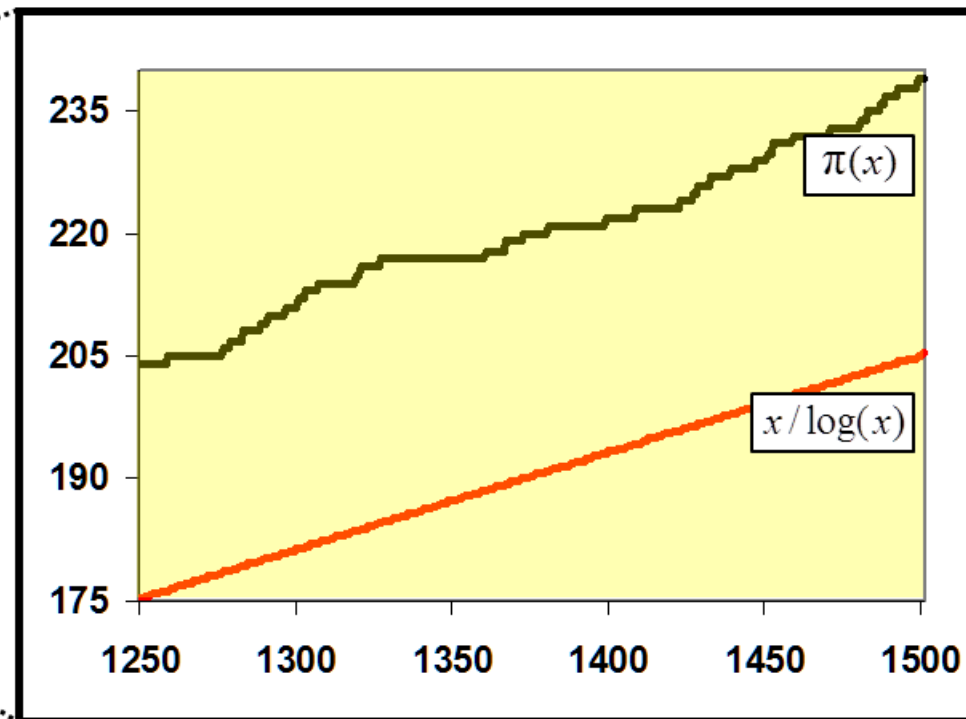
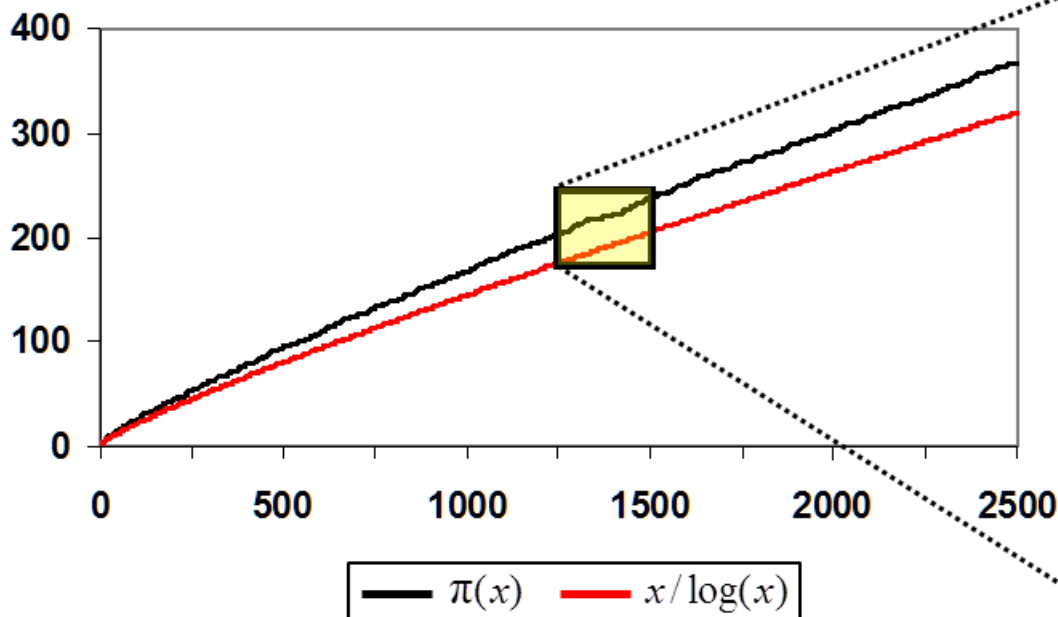




# THEOREM OF THE DAY



**The Prime Number Theorem** *The number of primes not exceeding  $x$  is asymptotic to  $x/\log x$ .*



The number of primes not exceeding a real number  $x$  is usually denoted by  $\pi(x)$ . The values of  $\pi(x)$  are plotted in the graph above left, for integer values of  $x$  up to 2500, as the upper, black line. The lower, red line shows the value of  $x/\log(x)$  for the same range of  $x$  (note that logarithms in number theory are generally taken with  $e = 2.71828\dots$  as the base). The black line appears to follow the red line quite smoothly but, when magnified as in the right-hand graph, it is seen to climb jerkily, like a staircase, with a plateau before each new prime appears. Note the meaning of ‘asymptotic’: as the  $x$  values get larger the black and red lines grow further apart but the *ratio* of the values becomes closer and closer to 1. Thus, to 2 decimal places of accuracy, the value of  $\pi(1500) \ln(1500)/1500$  is 1.17 while the value of  $\pi(2500) \ln(2500)/2500$  is 1.15. The approach is not monotonic: at  $x = 1250$  the ratio is 1.16 to 2 decimal places. And it is rather slow: by  $x = 10^9$  our ratio is still above 1.05. It follows from the prime number theorem that the  $n$ -th prime number is approximately  $n \log n$  (although, again, the accuracy does not increase monotonically: the 1000th and 10000th primes are 7919 and 104729 and these are approximated to an accuracy of, respectively, 90% and 87%). Correspondingly, the probability that a number  $n$  is prime can be approximated to  $1/\log(n)$ .

The prime number theorem was conjectured in the 1790s by Adrien-Marie Legendre and, independently, by the teenaged Gauss. Proofs by Hadamard and de la Vallée Poussin came one hundred years later, again independently. Riemann’s 1859 study of  $\pi(x)$  and its connection with the Riemann zeta function was fundamental in these proofs which led on to the still greater quest, now well into its second century, to prove the Riemann Hypothesis.

**Web link:** [www.mat.utfsm.cl/scientia](http://www.mat.utfsm.cl/scientia): Fine and Rosenbergers’s fine **historial survey** in **Vol. 20**. Find  $\pi(x)$  for  $x \leq 10^{13}$  at [primes.utm.edu/nthprime](http://primes.utm.edu/nthprime).

**Further reading:** *The Prime Number Theorem* by G.J.O. Jameson, Cambridge University Press, 2003.

