THEOREM OF THE DAY

The Eratosthenes–Legendre Sieve *Let $\pi(x)$ denote the number of primes not exceeding x, and $P(x)$ denote the product of all primes not exceeding x. Then*

$$\pi(x) - \pi(\sqrt{x}) + 1 = \sum_{d \mid P(\sqrt{x})} \mu(d) \left\lfloor \frac{x}{d} \right\rfloor,$$

where $\mu(n)$ is the Möbius function defined for positive integers n by

$$\mu(n) = \begin{cases} (-1)^r & \text{if } n \text{ is a product of } r \text{ distinct primes (with } r = 0 \text{ if } n = 1) \\ 0 & \text{if } n \text{ has a square factor.} \end{cases}$$

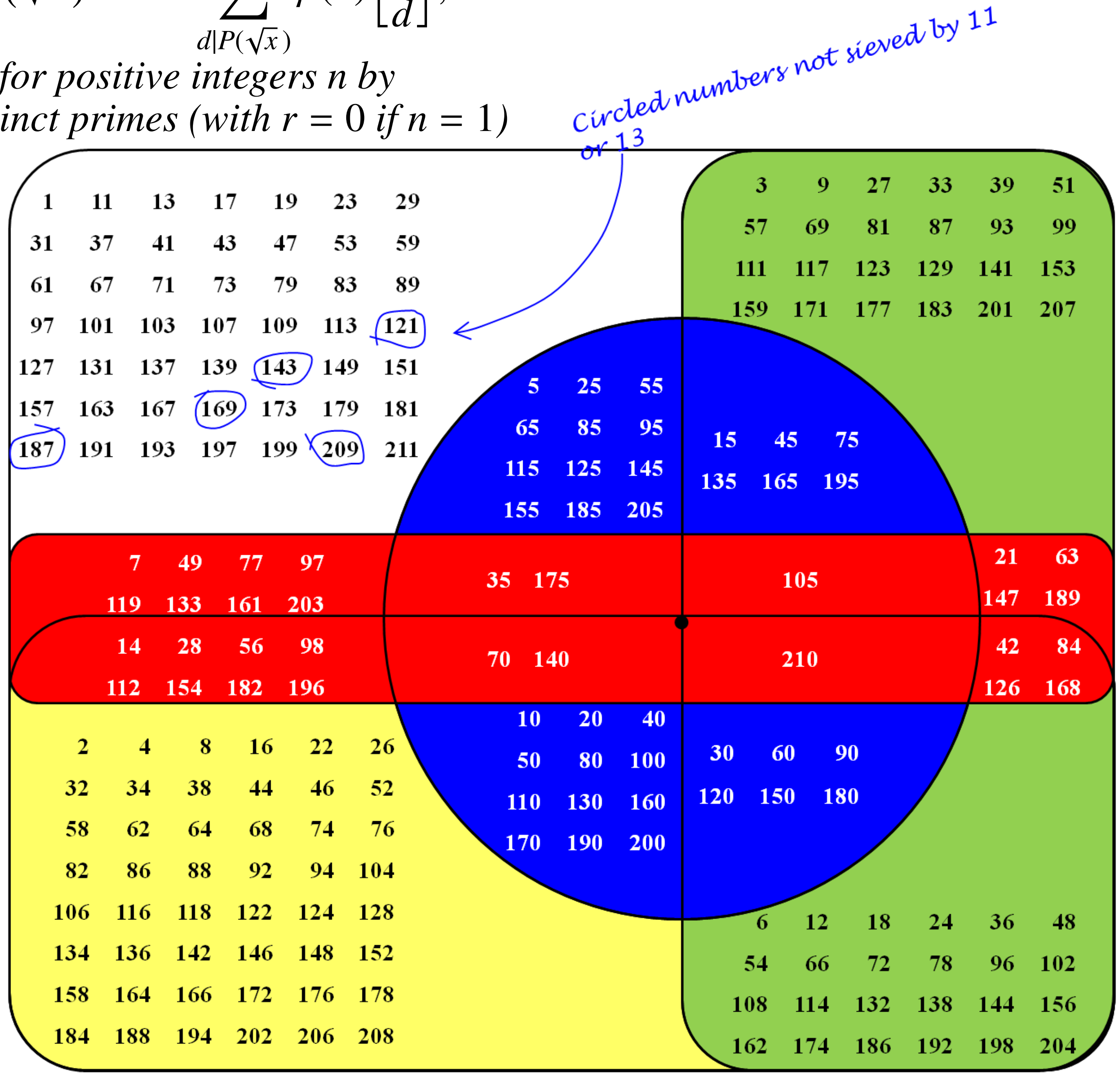
Our summation is over all positive integers d which divide $P(\sqrt{x})$; but only $d \leq x$ count since $\lfloor x/d \rfloor$, the greatest integer not exceeding x/d, becomes zero when $d > x$.

How might we count the primes up to $211 = 1 + 2 \times 3 \times 5 \times 7$? A first approximation is to count all the integers from 1 to 211 which are excluded from the shaded regions of the 4-set Venn diagram on the right: bottom = multiples of 2; right = multiples of 3; circle=multiples of 5; central = multiples of 7. Inclusion-exclusion 'sieves out' products of just the first four primes (# denotes 'number of'):

			211
−	# multiples of 2, 3, 5, 7	$\left\lfloor \frac{211}{2} \right\rfloor + \ldots + \left\lfloor \frac{211}{7} \right\rfloor$	247
+	# multiples of $2 \times 3, 2 \times 5, \ldots, 5 \times 7$	$\left\lfloor \frac{211}{6} \right\rfloor + \ldots + \left\lfloor \frac{211}{35} \right\rfloor$	101
−	# multiples of $2 \times 3 \times 5, \ldots, 3 \times 5 \times 7$	$\left\lfloor \frac{211}{30} \right\rfloor + \ldots + \left\lfloor \frac{211}{105} \right\rfloor$	17
+	# multiples of $2 \times 3 \times 5 \times 7$	$\left\lfloor \frac{211}{210} \right\rfloor$	1
	Total (as shown in top-left of Venn diagram):		49

We have an estimate $\pi(211) - 4 + 1 \approx 49$, compensating for our four primes which have been sieved out, and counting the non-sieved non-prime 1. Legendre's version of Eratothenes' sieve is exact, extending inclusion-exclusion up to $\pi(\sqrt{211}) = 6$, the maximum possible, giving $\pi(211) - 6 + 1 = 42$. The Möbius function cleverly converts the alternating double sum of inclusion-exclusion into a single sum.

Eratosthenes, around 100BC, is credited with inventing the method of listing primes by sieving. Its adaptation by Legendre in 1808 to count primes is conceptually behind all modern sieve-based methods in number theory.



Web link: press.princeton.edu/titles/8585.html: Ch. 1 of Glyn Harman's *Prime-Detecting Sieves*.
Further reading: *Sieve Methods* by Heini Halberstam and Hans-Egon Richert, Dover, 2011.