

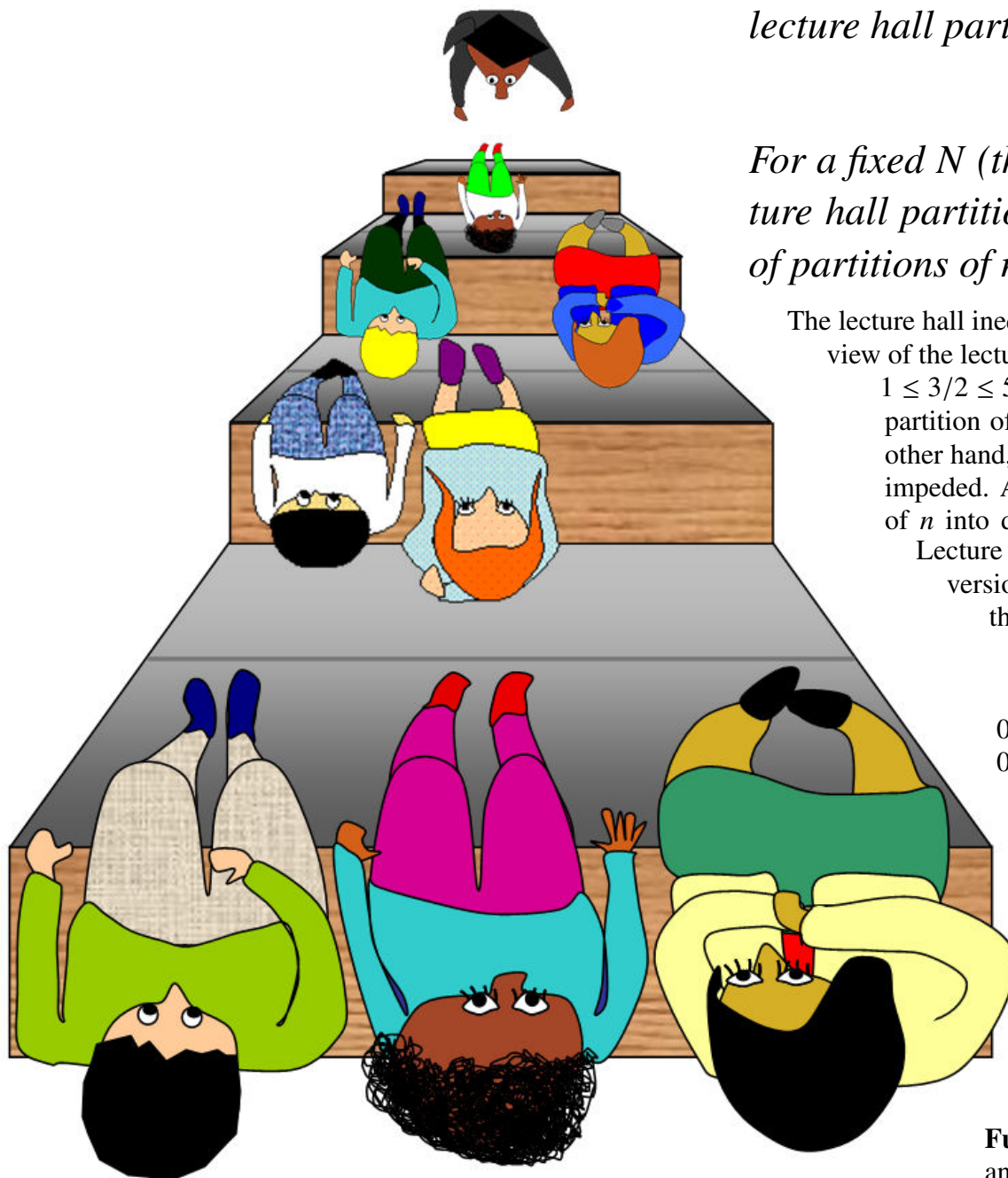


# THEOREM OF THE DAY

The Lecture Hall Partition Theorem A list  $(h_1, h_2, \dots, h_N)$  of non-negative integers is said to be a lecture hall partition if

$$0 \leq \frac{h_1}{1} \leq \frac{h_2}{2} \leq \dots \leq \frac{h_N}{N}.$$

For a fixed  $N$  (the length of the lecture hall), the number of lecture hall partitions of any positive integer  $n$  equals the number of partitions of  $n$  into odd parts smaller than  $2N$ .



The lecture hall inequalities ensure that each row in a tiered lecture hall will have a clear view of the lecturer. The hall shown on the left has rows at height 1, 3, 5 and 7; since  $1 \leq 3/2 \leq 5/3 \leq 7/4$ , the partition of 16 given by  $1 + 3 + 5 + 7$  is a lecture hall partition of length  $N = 4$ . The partition of 16 given by  $1 + 4 + 5 + 6$ , on the other hand, represents a lecture hall in which the third and fourth row views are impeded. A famous partition identity of Leonhard Euler says that the partitions of  $n$  into distinct parts are equinumerous with those into odd parts; because Lecture Hall Partitions permit no equal (non-zero) parts, today's theorem is a version of Euler's identity with restricted maximum odd parts. If we allow the lecture hall to become infinitely long (removing the limit on  $N$ ) we recover Euler's identity in the limit. Here are the 21 lecture halls of length  $N = 4$  partitioning  $n = 16$  (note, that zeros are allowed):

0,0,0,16 0,0,1,15 0,0,2,14 0,0,3,13 0,1,2,13 0,0,4,12 0,1,3,12  
0,0,5,11 0,1,4,11 0,2,3,11 0,0,6,10 0,1,5,10 0,2,4,10 1,2,3,10  
0,1,6,9 0,2,5,9 1,2,4,9 0,2,6,8 0,3,5,8 1,2,5,8 1,3,5,7

The idea of a lecture hall partition arose unexpectedly in Kimmo Eriksson's work on Coxeter groups. He and Mireille Bousquet-Mélou proved the above (apparently difficult) identity and several deep generalisations in two influential papers in the late 1990s.

**Web link:** [www-igm.univ-mlv.fr/~fpsac/FPSAC08/fpsac08.html](http://www-igm.univ-mlv.fr/~fpsac/FPSAC08/fpsac08.html): the 'List of Presentations' has a fine (3MB) talk by Carla Savage. (There is also a talk by Bousquet-Mélou although not on partitions).

**Further reading:** *Integer Partitions, 2nd revised ed.* by George E. Andrews and Kimmo Eriksson, Cambridge University Press, 2004, chapter 9.

