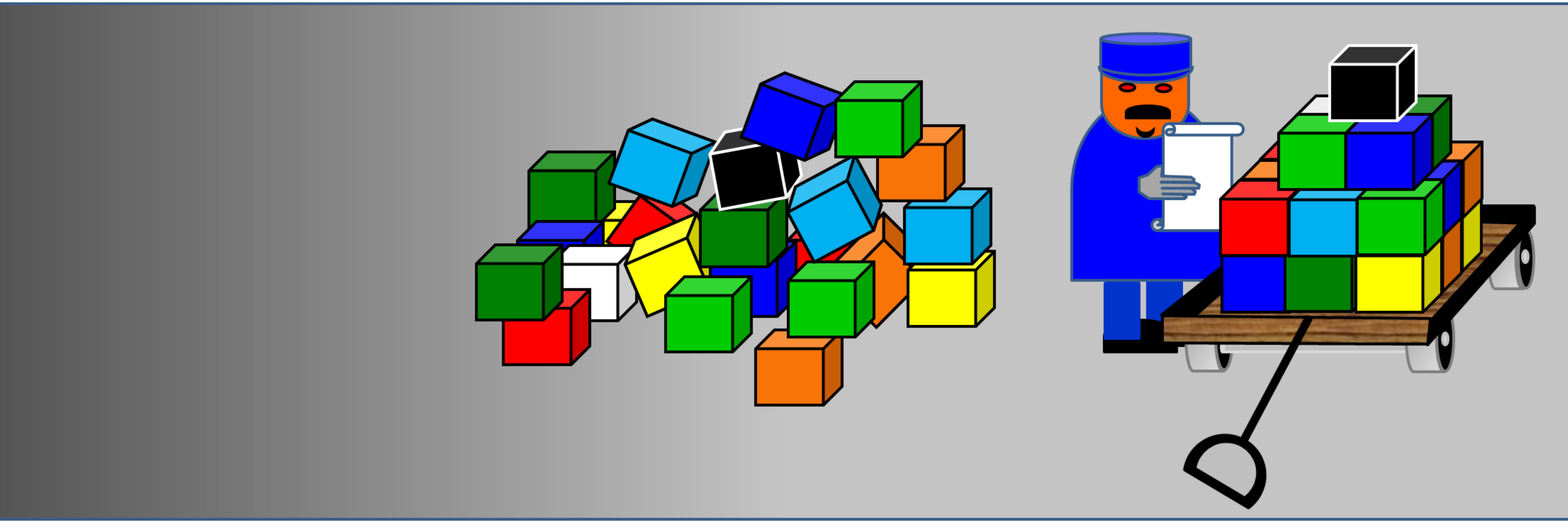
THEOREM OF THE DAY

Lagrange's Four-Squares Theorem *Any non-negative integer, n, may be written as a sum of four squares:*

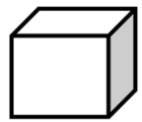
$$n = w^2 + x^2 + y^2 + z^2,$$

where w, x, y and z are non-negative integers (some of which may be zero.)



The Mathematical Porter insists on piling boxes on his trolley in a pyramid of square layers to a height of at most four. Lagrange's theorem says this may be accomplished no matter how many boxes the porter has. Here, the pyramid illustrates $23 = 3^2 + 3^2 + 2^2 + 1^2$.
But now the porter finds he has overlooked a box! How will he restack 24 boxes on his trolley?

The result was known to Diophantus of Alexandria and was first explicitly asserted by Bachet, who translated Diophantus's *Arithmetica* into Latin in 1621. Its proof required a hundred and fifty years of work by modern mathematicians, culminating in Lagrange's complete proof of 1770. More generally, **Waring's Problem** (solved affirmatively but non-constructively in 1909 by Hilbert) asks if *any positive integer n can be written as a sum of at most a fixed number of k-th powers*. For instance, any non-negative integer can be written as a sum of 9 cubes; thus $23 = 2^3 + 2^3 + 1^3 + 1^3 + 1^3 + 1^3 + 1^3 + 1^3 + 1^3$. Actually, this is one of only two numbers requiring 9 cubes (the other being 239) and it is still unknown whether, for large enough integers, 6 cubes might be enough (8042 is the largest integer known to require 7 cubes).



Web link: www.maths.lancs.ac.uk/~jameson/foursquares.pdf

Further reading: *Elementary Number Theory* by Gareth Jones and Mary Jones, Springer, Berlin, 1998.