



THEOREM OF THE DAY

Euler's Product Formula for $\zeta(s)$ For any complex number s having real part $Re(s) > 1$,

$$\zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} + \dots = \left(1 - \frac{1}{p_1^s}\right)^{-1} \left(1 - \frac{1}{p_2^s}\right)^{-1} \left(1 - \frac{1}{p_3^s}\right)^{-1} \dots$$

where p_1, p_2, p_3, \dots are the prime numbers.

Starting point: $\zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} + \dots$ (1)

Take any prime p

$$\frac{1}{p^s} \times (1) : \zeta(s) \frac{1}{p^s} = \frac{1}{p^s} \left(1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} + \dots\right)$$
 (2)

giving: $\zeta(s) \frac{1}{p^s} = \frac{1}{p^s} + \frac{1}{(2p)^s} + \frac{1}{(3p)^s} + \frac{1}{(4p)^s} + \dots$ (3)

(1) - (3) : $\zeta(s) \left(1 - \frac{1}{p^s}\right) = 1 + \dots$ no $\frac{1}{(kp)^s}$ terms! (4)

Repeat the above process for every prime p . On the left-hand side we get:

$$\zeta(s) \left(1 - \frac{1}{p_1^s}\right) \left(1 - \frac{1}{p_2^s}\right) \left(1 - \frac{1}{p_3^s}\right) \dots = \zeta(s) \prod_p \left(1 - \frac{1}{p^s}\right)$$

All that will remain on the right is ... 1. Divide out:

$$\zeta(s) = \frac{1}{\prod_p \left(1 - \frac{1}{p^s}\right)} = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}$$

With his formula, Euler initiates an extended analysis which culminates in a proof that the sum of the reciprocals of the primes diverges. It was presented to the St. Petersburg Academy in 1737 and was published by them in 1744 (see the facsimile on the right, courtesy of the wonderful Euler Archive, and spot Euler's misprint!) The convergence properties of the formula were not studied until the next century and the extension to complex numbers is due to Riemann in his famous 1859 "Über die Anzahl der Primzahlen".

The screenshot shows the 'The Euler Archive' website interface. At the top, it says 'The Euler Archive' and 'A digital library dedicated to the work and life of Leonhard Euler'. Below this is a portrait of Euler. On the left is a navigation menu with options like 'Main Page', 'Navigation Help', 'Search archive by:' (with sub-options for Subject, Date, Publication, Index Number), 'Historical Information:' (with sub-options for 18th Century Europe, The Life of Euler, Contemporaries, Important Locations), and 'Archive Features:' (with sub-options for Translations, Correspondence, Further Reading). The main content area displays a facsimile of a handwritten manuscript page titled 'Theorema 8.' and 'CIRCA SERIES INFINITAS.' The text includes mathematical expressions for the sum of reciprocals of primes and a 'Demonstratio' section.

Facsimile of Euler's "Variae observationes circa series infinitas", entry E072 at eulerarchive.maa.org/

Web link: www.maths.tcd.ie/pub/Maths/Courseware/428/ (notes by Timothy Murphy, Euler's formula is discussed in **Primes-II**).
Further reading: *How Euler Did It* by C. Edward Sandifer, MAA, 2007, Chapter 33.

