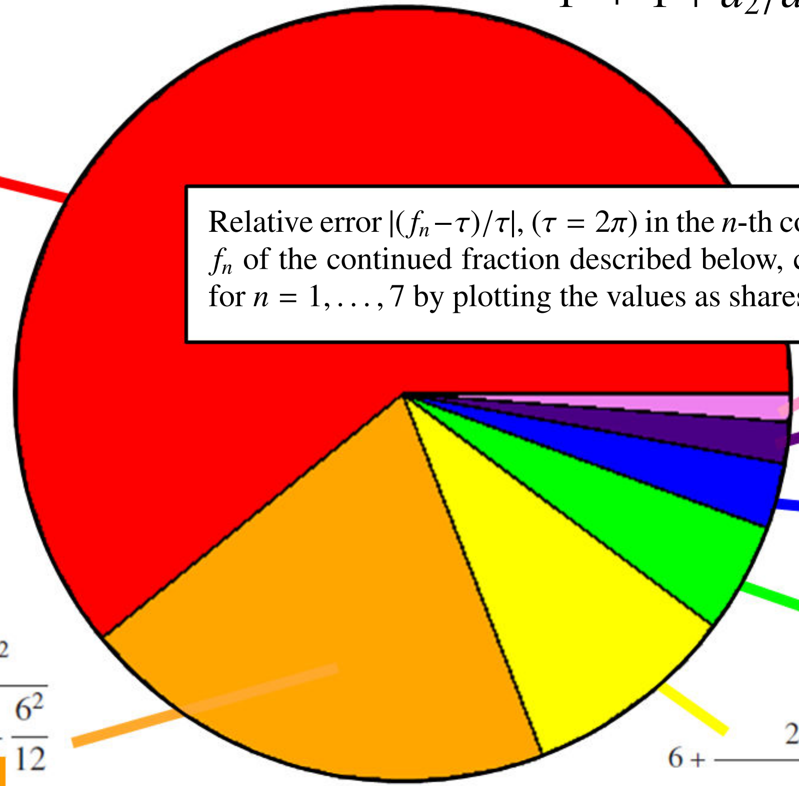
THEOREM OF THE DAY

Euler's Continued Fraction Correspondence *Let $(a_i)_{i\geq 0}$ be an infinite sequence of nonzero real or complex numbers. Let f_n denote the n-th partial sum of the sequence: $f_n = \sum_{i=0}^{n} a_i$. Then f_n is also the n-th convergent of the continued fraction*

$$a_0 + \frac{a_1}{1} + \frac{-a_2/a_1}{1 + a_2/a_1} + \ldots + \frac{-a_n/a_{n-1}}{1 + a_n/a_{n-1}} + \ldots.$$



Relative error $|(f_n-\tau)/\tau|$, $(\tau = 2\pi)$ in the n-th convergent f_n of the continued fraction described below, compared for $n = 1, \ldots, 7$ by plotting the values as shares of a pie.

$$6 + \cfrac{2^2}{12}$$ **6.333333333**

$$6 + \cfrac{2^2}{12 + \cfrac{6^2}{12}}$$ **6.266666667**

$$6 + \cfrac{2^2}{12 + \cfrac{6^2}{12 + \cfrac{10^2}{12}}}$$ **6.290476190**

$$6 + \cfrac{2^2}{12 + \cfrac{6^2}{12 + \cfrac{10^2}{12 + \cfrac{14^2}{12}}}}$$ **6.279365079**

$$6 + \cfrac{2^2}{12 + \cfrac{6^2}{12 + \cfrac{10^2}{12 + \cfrac{14^2}{12 + \cfrac{18^2}{12}}}}}$$ **6.285425685**

$$6 + \cfrac{2^2}{12 + \cfrac{6^2}{12 + \cfrac{10^2}{12 + \cfrac{14^2}{12 + \cfrac{18^2}{12 + \cfrac{22^2}{12}}}}}}$$ **6.281762682**

$$6 + \cfrac{2^2}{12 + \cfrac{6^2}{12 + \cfrac{10^2}{12 + \cfrac{14^2}{12 + \cfrac{18^2}{12 + \cfrac{22^2}{12 + \cfrac{26^2}{12}}}}}}}$$ 6.284143634

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An infinite summation due to the 15th century Indian scholar **Kerala Gargya Nilakantha**:

$$\frac{\tau - 6}{4} = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k(2k+1)(2k+2)} = \frac{1}{1.3.4} - \frac{1}{2.5.6} + \frac{1}{3.7.8} - \frac{1}{4.9.10} + \ldots.$$

If we apply Euler's correspondence to Nilakantha's series with $a_i = (-1)^{k-1}/k(2k+1)(2k+2)$ then we get $\frac{\tau-6}{4} = \frac{1}{12} + \frac{1^2.3^2.4^2}{12.2^2} + \frac{2^2.5^2.6^2}{12.3^2} + \frac{3^2.7^2.8^2}{12.4^2} + \ldots$, giving $\tau = 6 + \frac{4}{12} + \frac{1^2.3^2.2^2.\not{2}^2}{12.\not{2}^2} + \frac{\not{2}^2.5^2.2^2.\not{3}^2}{12.\not{3}^2} + \frac{\not{3}^2.7^2.2^2.\not{4}^2}{12.\not{4}^2} + \ldots = 6 + \frac{2^2}{12} + \frac{6^2}{12} + \frac{10^2}{12} + \ldots$, whose convergents are explored in the above pie chart.

Leonhard Euler discovered this correspondence in 1748. The above application to τ (in a π version) was given by Douglas Bowman as an alternative derivation of a continued fraction published by Jerome Lange in 1999.

Web link: www.math.binghamton.edu/dikran/478/Ch7.pdf. More on the τ continued fraction: www.maths.qmul.ac.uk/~whitty/Oxford/Tauvpi/.

Further reading: *Handbook of Continued Fractions for Special Functions* by Annie Cuyt et al, Springer, 2008.