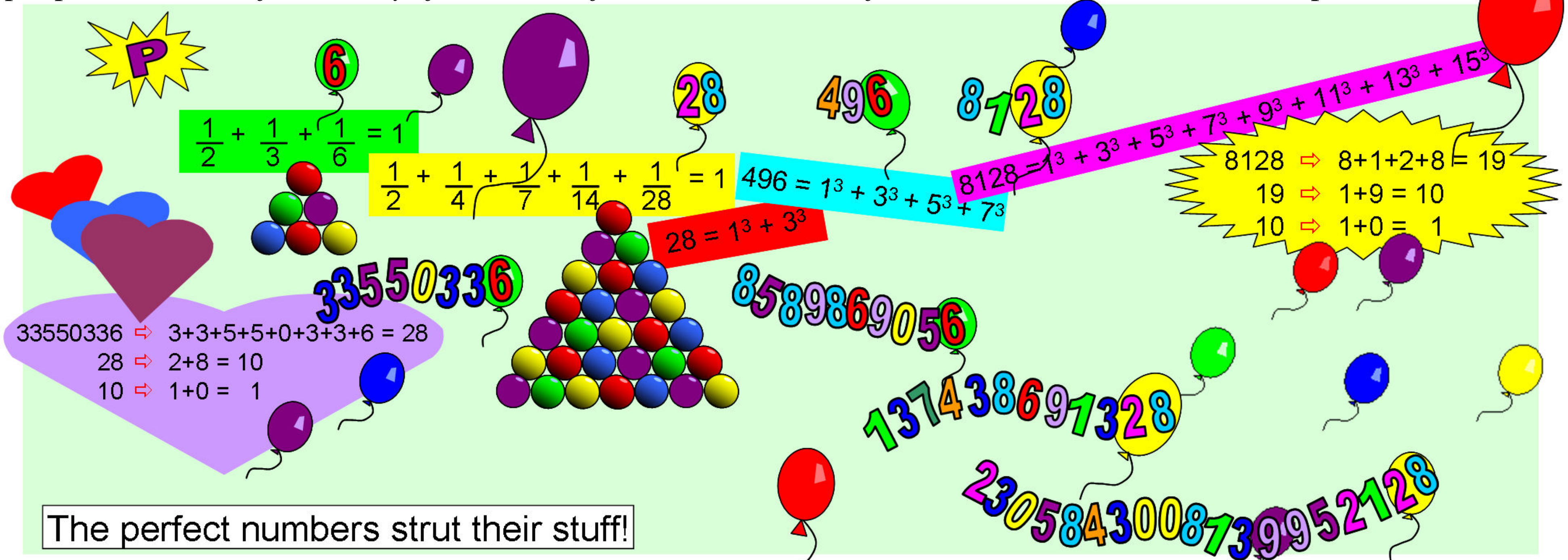
THEOREM OF THE DAY

The Euclid-Euler Theorem *An even positive integer is a perfect number, that is, equals the sum of its proper divisors, if and only if it has the form $2^{n-1}(2^n-1)$, for some n such that 2^n-1 is prime.*



The perfect numbers strut their stuff!

A prime of the form 2^n-1 must have n also prime, a fact sometimes known as the **Cataldi-Fermat Theorem**; it thus belongs to the so-called *Mersenne primes*, which by today's theorem are in one-to-one correspondence with the even perfect numbers, whose infinitude consequently remains unknown, since only 49 Mersenne primes are known, as of January 2016, to exist.

The even perfect numbers have many other striking properties, as illustrated above — they all end in 6 or 28; they have *digital root*, formed by repeatedly summing their decimal digits, equal to 1, which is also the sum of the reciprocals of their non-trivial divisors; they are triangular numbers (a summation of consecutive integers starting at 1); a perfect number $2^{n-1}(2^n-1)$ for an odd prime n, rewritten in the form $w^2\left(2w^2-1\right)$, is equal to the sum of the cubes of the first w odd numbers. The *odd* perfect numbers also have striking properties, so striking that Sylvester, in 1888, was led to say "a prolonged meditation on the subject has satisfied me that the existence of any one such — its escape, so to say, from the complex web of conditions which hem it in on all sides — would be little short of a miracle. Thus then there seems to be every reason to believe that Euclid's perfect numbers are the only perfect numbers which exist!" At any rate, it is known, as of 2012, that any odd perfect number must exceed 10^{1500}.

A proof of the fact that $2^{n-1}(2^n-1)$ is perfect when 2^n-1 is prime appeared as the climax (Prop. 36) to Book IX of Euclid's *Elements*. Leonhard Euler supplied the proof that these are the only even perfect numbers and also initiated the search for odd perfect numbers, whose non-existence remains unproven to this day.

Web link: www.math.dartmouth.edu/~jvoight/notes/perfelem.pdf offers no fewer than six proofs of the Euclid-Euler Theorem. The Sylvester quote is found here (page 6): www.emis.de/journals/INTEGERS/papers/d16/d16.pdf (560KB). Computer search updates: www.mersenne.org/.

Further reading: *Elementary Number Theory, 7th ed.,* by David M. Burton, MacGraw-Hill, 2005, chapter 10.