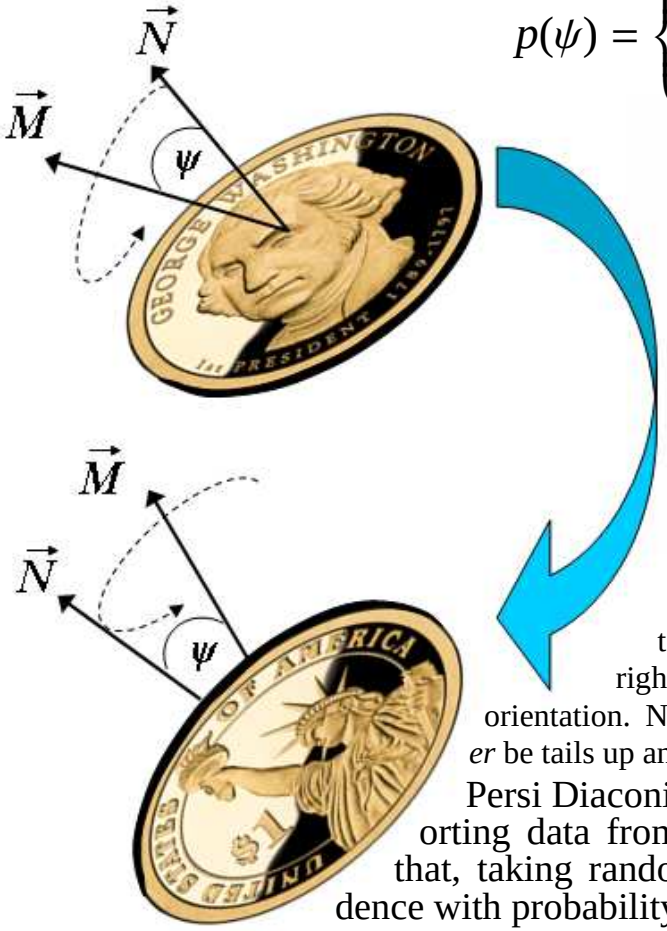
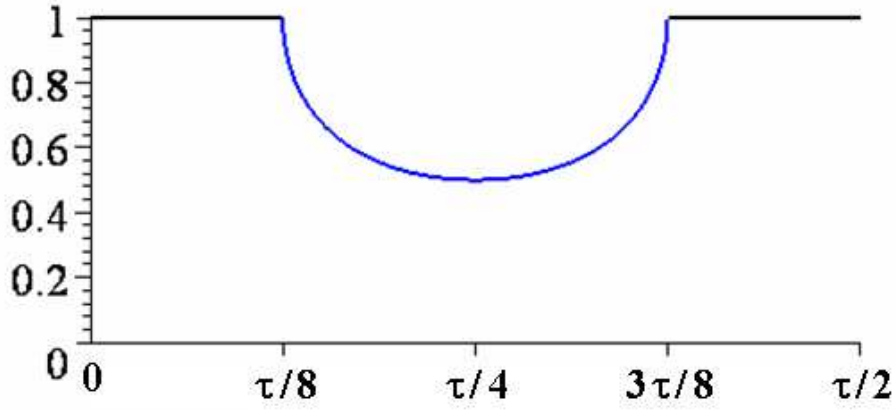
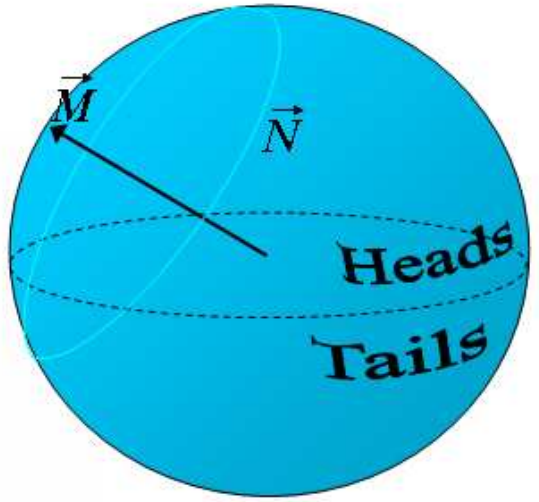
THEOREM OF THE DAY

The Diaconis–Holmes–Montgomery Coin Tossing Theorem *Suppose a coin toss is represented by: ω, the initial angular velocity; t, the flight time; and ψ, the initial angle between the angular momentum vector and the normal to the coin surface, with this surface initially 'heads up'. Consider the pair (ω, t) as a smooth, compactly supported random variable, and let the centre of its distribution tend to infinity in the positive orthant (corresponding to large spin and long flight time). Then with ψ fixed, the limiting probability $p(\psi)$ of the coin landing heads is given by*

$$p(\psi) = \begin{cases} \frac{1}{2} + \frac{2}{\tau}\sin^{-1}(\cot^2\psi) & \text{if } \frac{\tau}{8} < \psi < \frac{3\tau}{8} \ (\tau = 2\pi) \\ 1 & \text{if } 0 < \psi < \frac{\tau}{8} \text{ or } \frac{3\tau}{8} < \psi < \frac{\tau}{2}. \end{cases}$$







The key advance encapsulated in this theorem is the inclusion of precession: the normal $\vec{N}$ to the coin surface rotates about the base of the angular momentum vector $\vec{M}$ as shown on the left. The angle ψ between the two vectors is shown to remain constant. The effect on the orientation of the coin can be viewed in terms of a sphere, above right, whose equator is the flat 'heads up' plane. Only if ψ is sufficiently large will the coin ever flip over into the tails orientation. Note, in particular, that if $\psi = \tau/4$ then the coin spends an equal time heads up and tails up; if $\psi < \tau/8$ it will *never* be tails up and a heads outcome is a certainty. This is in accordance with the function $p(\psi)$ whose graph is plotted above centre.

Persi Diaconis, Susan Holmes and Richard Montgomery published this analysis in 2007, together with supporting data from detailed experiments. Although the heads bias seems conclusively demonstrated they remark that, taking random external factors into consideration: "for tossed coins, the classical assumptions of independence with probability 1/2 are pretty solid."

Web link: www.ams.org/samplings/math-history/happening-series#vol7: see "The Fifty-One Percent Solution"; the images of the US dollar coin are used courtesy of the United States Mint).

Further reading: *The Feynman Lectures on Physics: Definitive and Extended Edition* by Richard P. Feynman, Robert B. Leighton and Matthew Sands, Addison Wesley, 2005, vol. 1, chapter 18.