THEOREM OF THE DAY

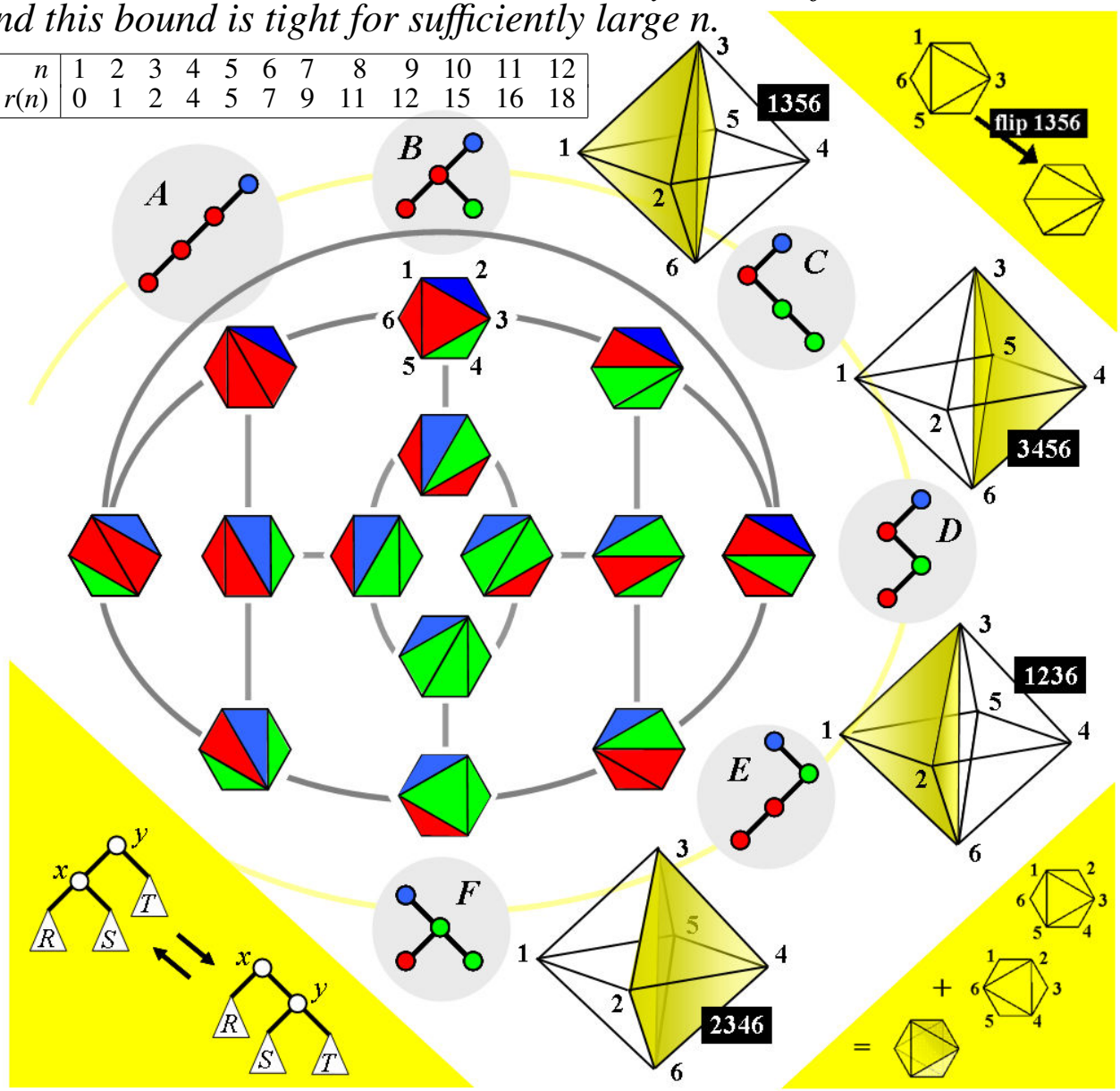
The Rotation Distance Bound *Any binary tree on n internal vertices, $n \geq 11$, may be transformed into any other using at most $2n - 6$ rotations; and this bound is tight for sufficiently large n.*

A binary tree consists of a *root vertex* joined to either zero or two *subtrees*, each of which is again a binary tree. Traditionally it is depicted as branching downwards with each vertex branching left (the *left child*) and right (*right child*). Vertices which have no subtrees are omitted from the drawing, in which only the *internal vertices* are shown. Six of the 14 distinct binary trees on four internal vertices are shown on the right, labelled A–F. The colours, red for left and green for right, are for illustrative purposes only.

Suppose a vertex y has left and right subtrees, with x being the root of the left subtree. A *rotation* at x moves x into y's place and y to the place of its right child, see the illustration, bottom left, in the yellow triangle. The left subtree of x and the right subtree of y (R and T, respectively, in the picture) are maintained, but the right subtree of x (i.e. S) becomes the left subtree of y. A rotation at the root of a right subtree, reversing the process, is also shown.

It is well-known that any binary tree may be transformed into any other on the same number of vertices by a series of rotations; thus B is transformed into F in the illustration; using, moreover, the least possible number of rotations. In 1988 Daniel Sleator, Robert Tarjan and William Thurston put an upper bound on the maximum distance $r(n)$, in number of rotations, between two trees with n internal vertices, by mapping such trees to triangulations of the $(n+2)$-gon. Trees A–F are shown adjacent to their corresponding triangulations here. Under the mapping rotations become 'flips': see the upper right yellow triangle; and some simple flip counting yields the upper bound $r(n) \leq 2n - 6$. Glueing together two polygons creates a polyhedron (an octagon in our case: bottom right triangle); a sequence of rotations becomes a 'triangulation' of the polyhedron (in our case using tetrahedra). And now an inspired switch from Euclidean to hyperbolic geometry produces, when n is sufficiently large, examples of triangulations which correspond to sequences of rotations meeting the $2n - 6$ bound.

n	1	2	3	4	5	6	7	8	9	10	11	12
$r(n)$	0	1	2	4	4	5	7	9	11	12	15	16	18



Lionel Pournin proved in 2012 that this famous 1988 bound is in fact achieved for all $n \geq 11$, see gilkalai.wordpress.com/2012/12/30/.

Web link: www.math.unicaen.fr/~dehornoy/Papers/Dhw.pdf. The original 1988 paper is here: www.cs.cmu.edu/~sleator/papers/Rotation-Distance.htm.

Further reading: T. Cormen, C. Leiserson, R. Rivest and C. Stein, Introduction to Algorithms, 3rd edition, MIT Press 2009, chapter 13.