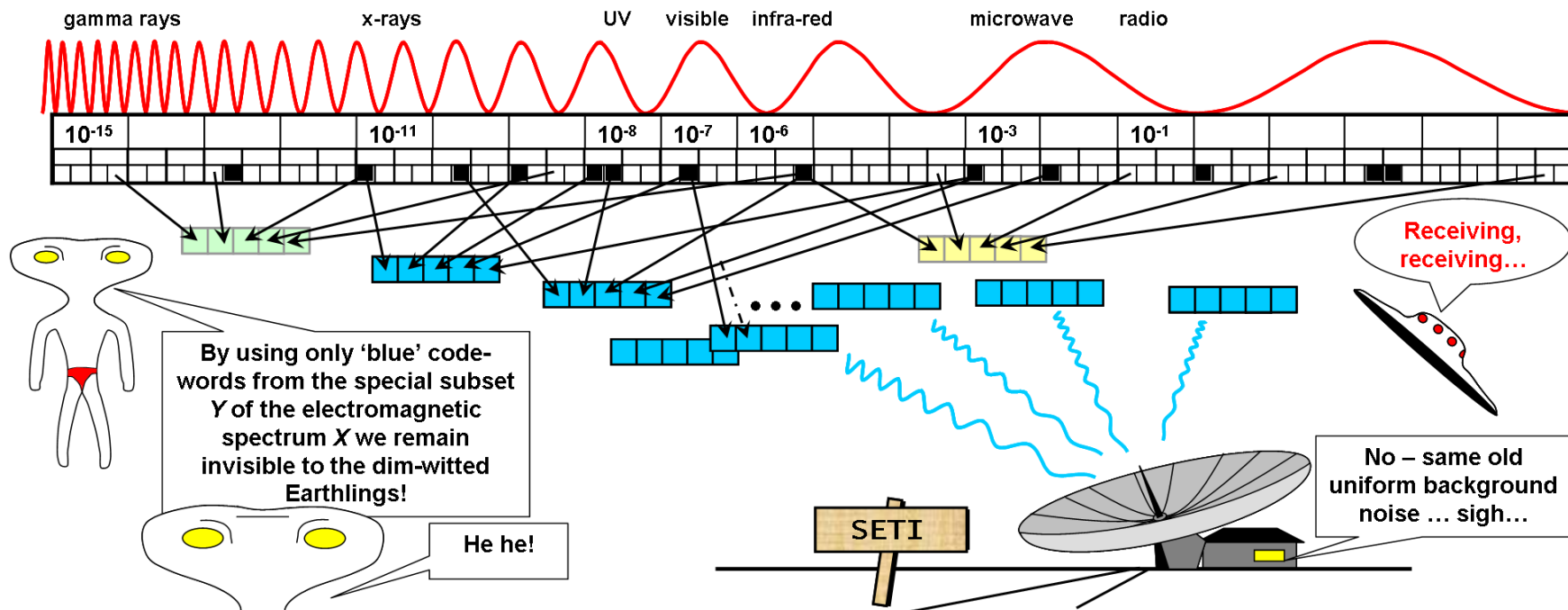




# THEOREM OF THE DAY

**Gödel's Second Incompleteness Theorem** *Let  $T$  be a recursively enumerable axiomatisation of number theory. Then the consistency of  $T$  cannot be proved as a theorem of  $T$ , i.e. "no falsehood can be derived from the axioms of  $T$ " cannot be derived from the axioms of  $T$ .*



In 1977, 35 years after Gödel proved his First Incompleteness Theorem, Jeff Paris and Leo Harrington were the first to show that it was more than a peculiarity of mathematical logic by exhibiting a 'natural' theorem which could be stated but not proved in Peano arithmetic (that is, with  $\forall$  and  $\exists$  allowed for variables but not for sets). Imagine that aliens want to avoid the attentions of SETI (Search for Extraterrestrial Intelligence). They communicate using the electromagnetic spectrum (EMS) but must do so without betraying unusual patterns of activity to SETI's scanners. Divide the EMS into a set  $X$  of frequencies and transmit codewords of length  $k$  from  $X$ . Suppose that SETI's technology is too crude to read the code — they can merely distinguish  $m$  different types (colours) of codeword. At the same time as Gödel proved his theorems, a classic theorem of F.P. Ramsey appeared which said (in effect) that the aliens can always divide the EMS into a large enough set  $X$  that there is a subset  $Y$  of  $X$ , as large as they need, with all  $k$ -length words from  $Y$  having the same colour. Paris–Harrington shows (in effect) that you can require, in addition, that the larger the set  $Y$ , the further along the EMS it must be located. But this stronger theorem, although still stated using Peano arithmetic, can no longer be proved from its axioms.

Ramsey's theorem is provable in Peano arithmetic. Paris and Harrington demonstrated that the truth of their subtle variation implied the consistency of Peano arithmetic. Therefore, by Gödel's Second Incompleteness Theorem, their result, provable from the axioms of set theory, cannot be proved from those of Peano arithmetic.

**Web link:** Joel Spencer's 1984 article at [www.maa.org/programs/maa-awards/writing-awards/paul-halmos-lester-ford-awards](http://www.maa.org/programs/maa-awards/writing-awards/paul-halmos-lester-ford-awards).

**Further reading:** *An Introduction to Gödel's Theorems* by Peter Smith, Cambridge University Press, 2nd edition, 2013.

