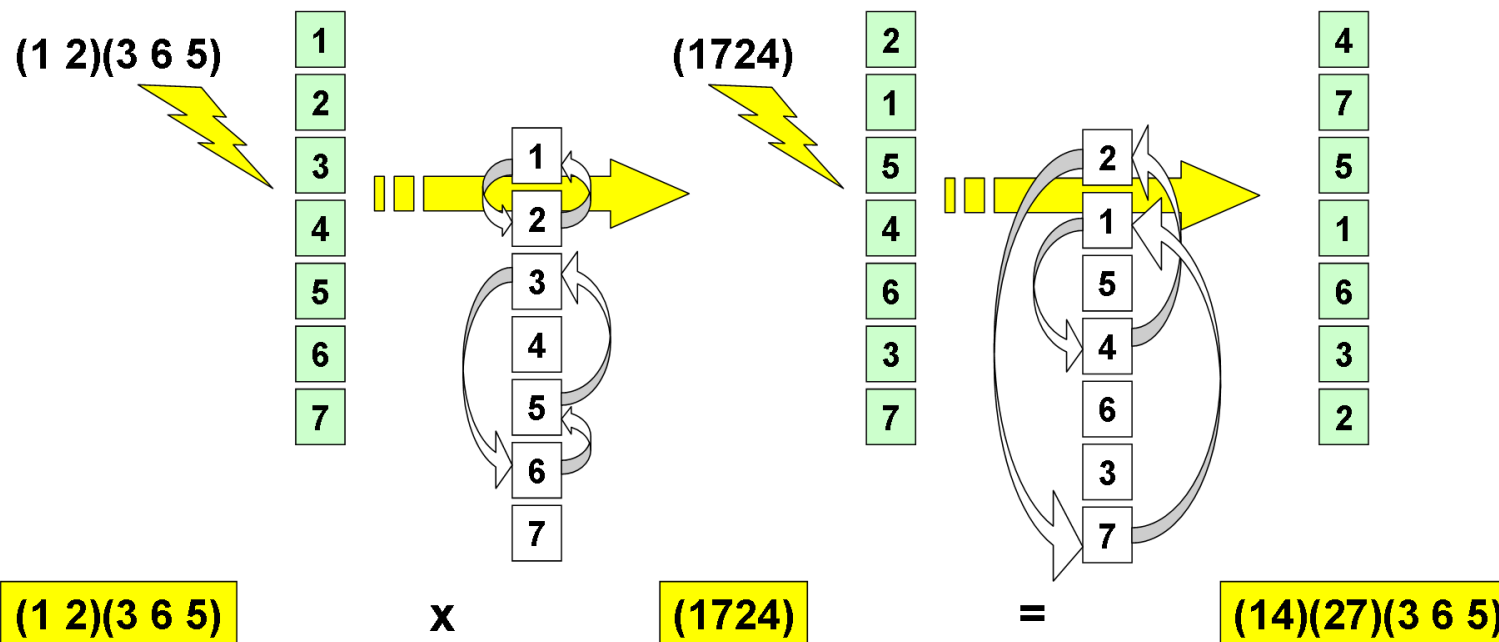




# THEOREM OF THE DAY

**The Orbit Counting Lemma** Let  $G$  be a finite permutation group on set  $\Omega$ . For  $g \in G$ , let  $\text{fix}(g)$  be the number of points of  $\Omega$  not moved by  $g$ . Then

$$\text{number of orbits of } G \text{ on } \Omega = \frac{1}{|G|} \sum_{g \in G} \text{fix}(g).$$



Here  $\Omega = \{1, 2, 3, 4, 5, 6, 7\}$ . Permutations on  $\Omega$  can be thought of as rearrangements of the contents of a line of 7 boxes, one for each element of  $\Omega$ . A combination, or *product*, of two permutations is shown here; the first has  $\text{fix}((1\ 2)(3\ 6\ 5)) = 2$ , since only the fourth and seventh boxes' contents are unmoved, while  $\text{fix}((1\ 7\ 4\ 2)) = 3$ . We get a finite permutation group by starting with a collection of permutations, including the *identity* which moves no box's contents, and successively producing all possible finite products. The two permutations used here generate a group of 24 different permutations, each of which can either move the contents of boxes 3, 5 and 6 or the contents of boxes 1, 2, 4 and 7, but without interchanging between these two subsets, which thereby constitute the two orbits of the group. The summation is as follows:

$$\frac{1}{24}(7 + 5 + 3 + 3 + 3 + 5 + 3 + 3 + 4 + 2 + 0 + 0 + 0 + 2 + 0 + 0 + 4 + 2 + 0 + 0 + 0 + 2 + 0 + 0) = \frac{1}{24}(48) = 2.$$

The Orbit Counting Lemma is often attributed to William Burnside (1852–1927). His famous 1897 book *Theory of Groups of Finite Order* perhaps marks its first 'textbook' appearance but the formula dates back to Cauchy in 1845.

**Web link:** [undergraduate.csse.uwa.edu.au/units/CITS7209](http://undergraduate.csse.uwa.edu.au/units/CITS7209), see [Lecture 3](#) (0.3MB).

**Further reading:** *Permutation Groups* by P.J. Cameron, Cambridge University Press, 1999.

