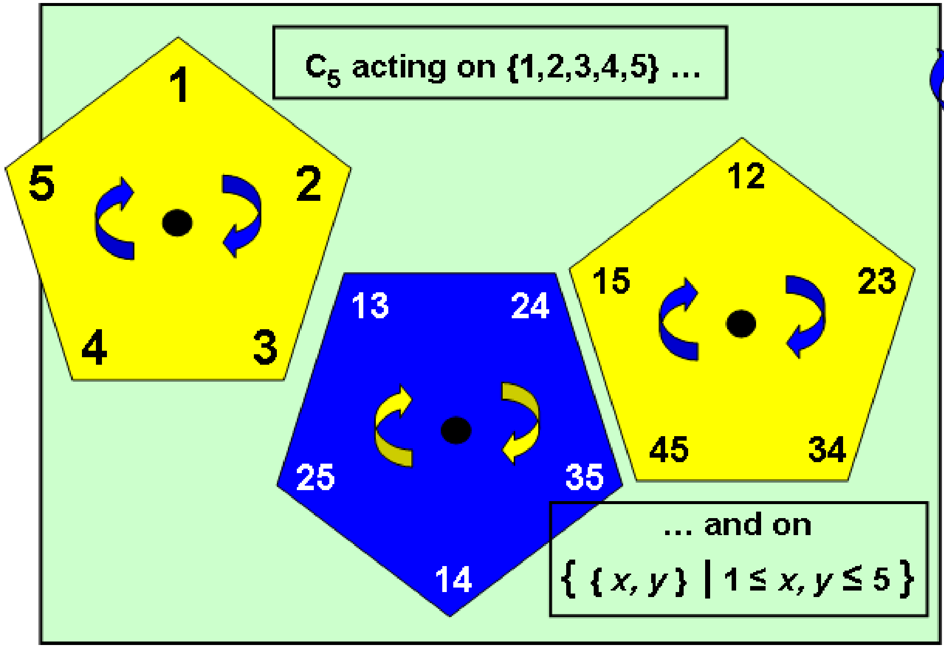
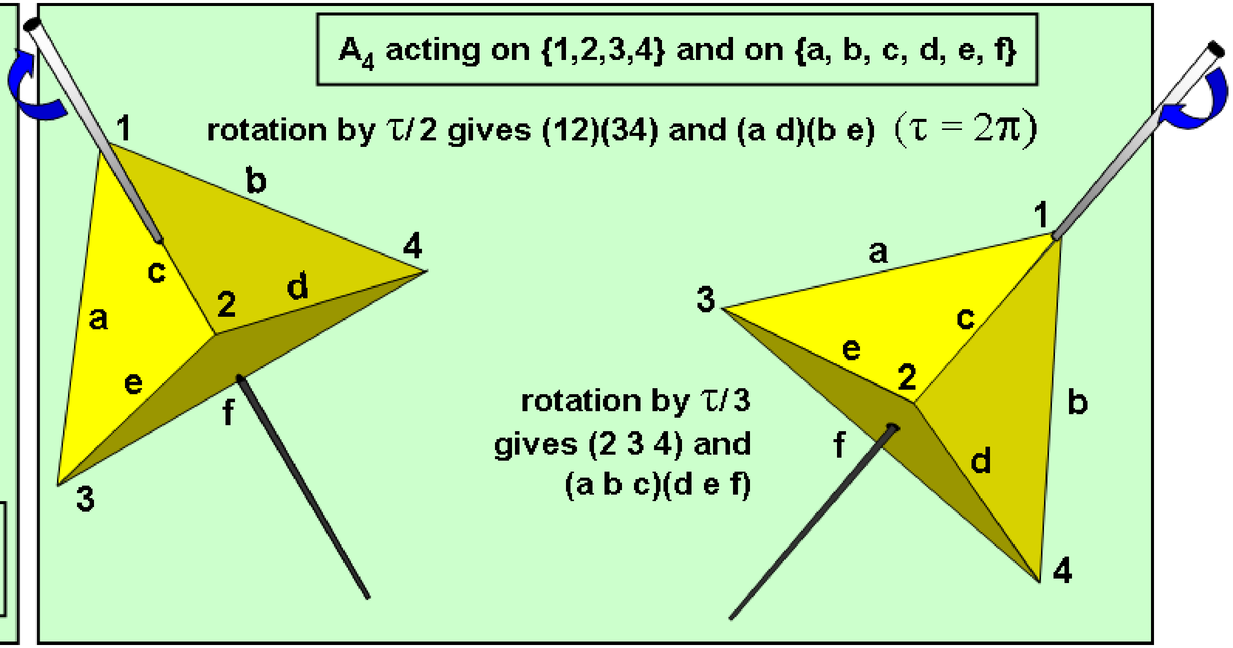
THEOREM OF THE DAY

Praeger's Theorem on Bounded Movement *Let G be a permutation group acting without fixed points on a set Ω. Denote by m the maximum size of any subset $\Gamma \subseteq \Omega$ whose image under some group element is disjoint from Γ; and suppose that m is finite. Then Ω is a finite set; the number t of G orbits is at most $2m-1$; each orbit has length at most $3m$; and $|\Omega| \leq 3m+t-1 \leq 5m-2$.*





The two examples above will serve to illustrate a couple of points of interest. On the left we see that m depends not only on the group G but also on its action; the cyclic group C_5 acts transitively on the pentagon: any point may be rotated to any other. The action has bounded movement with $m = 2$ since any three points will intersect with themselves under rotation. If instead C_5 acts on the ten unordered pairs of points then $m = 4$ (e.g., $\Gamma = \{12, 13, 34, 35\}$ becomes disjoint from itself under rotation). This latter action has two orbits and we can check that indeed $|\Omega| = 10 \leq 3m + t - 1 = 12 + 2 - 1 = 13$, as ordained by the theorem. The bound of $|\Omega| \leq 5m - 2$ becomes important when we do not know the value of t. Subsequent work by Praeger with Peter M. Neumann reduced it to $|\Omega| \leq (9m - 3)/2$, exhibiting moreover infinitely many examples of groups which attain this new bound. Notice, however, that our transitive C_5 action fails to meet the length bound of $3m$ on its single orbit.

The same is true for the transitive point action of the rotational symmetry group of the tetrahedron, isomorphic to the alternating group A_4 as shown above right. We have $m = 2$ and $|\Omega| = 4 < 3m = 6$. However, the transitive action of A_4 on the six edges of the tetrahedron again has $m = 2$ and does attain the bound; such transitive actions have been completely determined by A. Gardiner, Avinoam Mann and Praeger.

This 1991 theorem of Cheryl Praeger links back to work by BH Neumann in the 1950s. It has opened up a new line of enquiry for permutation group theorists, some of this work being alluded to above.

Web link: www.birs.ca/events/2009/5-day-workshops/09w5030 (click on 'final report')

Further reading: *Permutation Groups* by P.J. Cameron, Cambridge University Press, 1999.