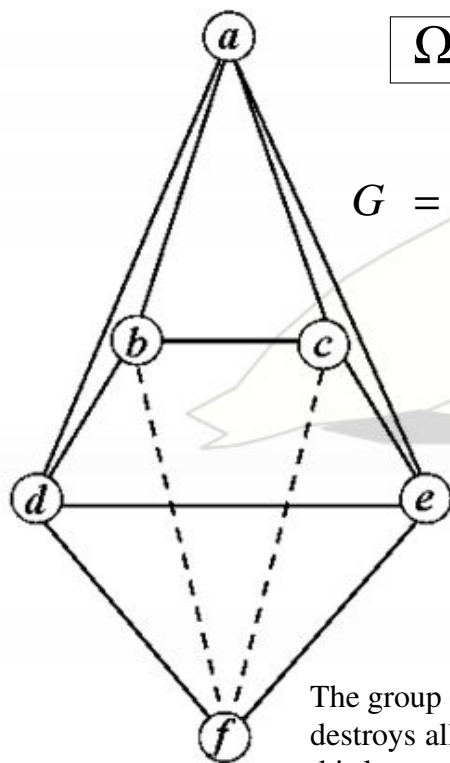




THEOREM OF THE DAY

The Cameron–Fon-Der-Flaass IBIS Theorem *Let G be a permutation group acting on a set Ω . Then the following are equivalent:*

1. all irredundant bases of G have the same size;
2. the irredundant bases of G are preserved by re-ordering;
3. the irredundant bases of G form the bases of a matroid.



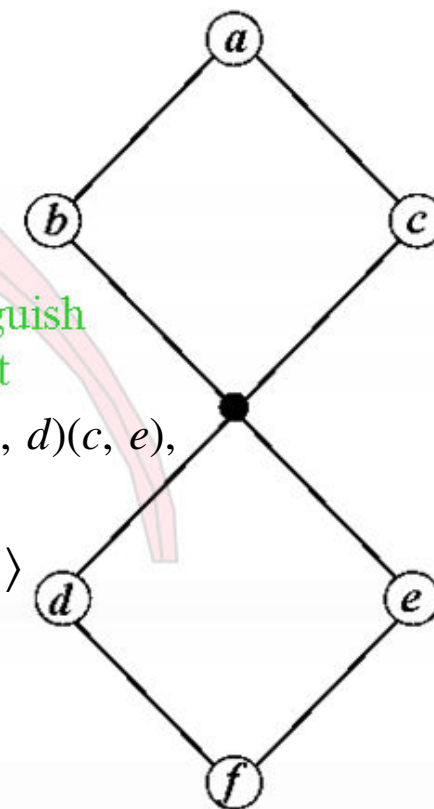
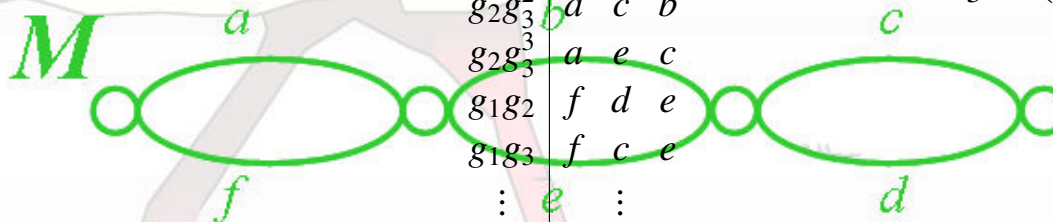
$$\Omega = \{a, b, c, d, e, f\}$$

$$G = \langle g_1 = (a, f), \\ g_2 = (b, d)(c, e), \\ g_3 = (b, c, e, d) \rangle$$

| G | 1 | a | b | c |
|------------|---|-----|----------|-----|
| | | a | b | c |
| g_1 | | f | b | c |
| g_2 | | a | d | e |
| g_3 | | a | c | e |
| g_3^2 | | a | e | d |
| g_3^3 | | a | d | b |
| g_2g_3 | | a | b | d |
| $g_2g_3^2$ | | a | c | b |
| $g_2g_3^3$ | | a | e | c |
| g_1g_2 | | f | d | e |
| g_1g_3 | | f | c | e |
| \vdots | | e | \vdots | |

← The action of a group on a base is sufficient to distinguish each group element

$$H = \langle h_1 = (a, f)(b, d)(c, e), \\ h_2 = (b, c), \\ h_3 = (d, e) \rangle$$



The group G represents the 16 symmetries of the irregular octahedron shown on the left. A **base** is a sequence of vertices which, when fixed, destroys all symmetry. Fixing vertex a removes g_1 , the vertical rotation; fixing b removes g_2 and g_3 but not $g_2g_3 = (c, d)$; fixing c removes this last symmetry: and $[a, b, c]$ is a base. Or we could finish by fixing d : so another base is $[a, b, d]$. In fact, in the graph M above-centre, any three edges which meet all three vertices, taken in any order, correspond to a base. Each of these bases is **irredundant**: each element of each sequence plays its part in removing additional symmetry. Moreover, M gives us a matroid: take any two bases B_1 and B_2 and they obey the ‘exchange property’ that we can replace some element in $B_1 \setminus B_2$ with an element in $B_2 \setminus B_1$ and again get a base. Not all groups have such well-behaved bases! The 8 symmetries of the figure on the right form the group H . Now $[a, b, d]$ is an irredundant base but the reordering $[b, a, d]$ is not irredundant: with b fixed, no remaining symmetries move a , so $[b, d]$ is a smaller base.

Peter Cameron and Dima Fon-Der-Flaass made this too-good-to-be-true link between group theory and combinatorics in 1995.

A group satisfying its three equivalent properties is called an IBIS group: it has ‘Irredundant Bases of Invariant Size’.

Web link: www.austms.org.au/Gazette/2005/May05/cohen.pdf; and see Elena Konstantinova’s tribute volume: math.nsc.ru/Archive/disk/memory/flaas.pdf (20MB).

Further reading: *Permutation Groups* by P.J. Cameron, Cambridge University Press, 1999, chapter 4.