



THEOREM OF THE DAY



The McIver–Neumann Half- n Bound Let Ω be a set of order n , $n \neq 3$, and let G be a permutation group acting on Ω . Then G may be generated by $\lfloor n/2 \rfloor$ elements.

$$S = \{(1\ 2)(3\ 4), (1\ 2\ 3\ 5)\}$$

Sift(σ)

while $\sigma \neq 1$ and not done **do**
 $i :=$ smallest row no. moved by σ
 $j := \sigma(i)$
 if M_{ij} not empty **then** $\sigma := \sigma \times M_{ij}^{-1}$
 else $M_{ij} := \sigma$ and done := true

Schreier-Sims(S)

Queue := S

while Queue not empty

$\sigma :=$ first permutation in Queue (thereupon removed)

if Sift(σ) updates M with σ' **then**

 add $\{M_{ij} \times \sigma', \sigma' \times M_{ij} \mid 1 \leq i < j \leq n, M_{ij} \neq \text{empty or } (\sigma')^{-1}\}$ to Queue

1	(12)(34)			
	1			
		1		
			1	
				1

1	(12)(34)			
	1		(2435)	
		1		
			1	
				1

1	(12)(34)		(1452)	
	1		(2435)	
		1		
			1	
				1

(about 40 steps, with maximum Queue size about 30)

The generation of a permutation group from a set of permutations is illuminated by the wonderful algorithm which Charles Sims derived from a 1927 lemma of Otto Schreier. In the illustration above, the algorithm is applied to a set of two permutations. The main work is done by the Sift routine, which first places $(1\ 2)(3\ 4)$ into row 1 column 2 cell in the 5×5 table M . The next permutation $\sigma = (1\ 2\ 3\ 5)$ is a candidate for the same cell since $\sigma(1) = 2$; since the cell is occupied by $(1\ 2)(3\ 4)$ we calculate $\sigma := \sigma \times M_{12}^{-1} = (1\ 2\ 3\ 5) \times (1\ 2)(3\ 4) = (2\ 4\ 3\ 5)$. The new cell M_{24} is now indicated for σ because now $\sigma(2) = 4$. Subsequently, the queue is augmented in the main algorithm by pre- and post-multiplying $(2\ 4\ 3\ 5)$ with each non-identity entry of M . The queue now begins with $(1\ 2)(3\ 4) \times (2\ 4\ 3\ 5) = (1\ 4\ 5\ 2)$ which Sift places directly into position $M_{1,4}$.

1	(12)(34)	(1345)	(1452)	(15)(24)
	1	(23)(45)	(2435)	(2534)
		1	–	–
			1	–
				1

Although the queue tends to grow rapidly it must eventually become empty; in the current example the algorithm terminates with table shown above, bottom-right. This table describes the generated group in the following way: the order of the group is the product of the numbers of permutations in each row, in this case $5 \times 4 \times 1 \times 1 \times 1 = 20$ (the group happens to be the Frobenius group of order 20, aka the Galois group of the polynomial $x^5 - 2$ over \mathbb{Q}); membership of the group is tested by running Sift on a candidate permutation σ which will be allocated to an empty table location if and only if it does not belong to the group. **Exercise:** try the algorithm with $\Omega = \{1, \dots, 8\}$ and $S = \{(1\ 2), (3\ 4), (5\ 6), (7\ 8)\}$.

It is easy to generate large groups from a very few permutations; $n/2$ permutations may sometimes be necessary as the Exercise above easily shows. But Annabelle McIver and Peter M. Neumann's 1987 theorem seems to be deep and mysterious: it rests on the Classification of the Finite Simple Groups and suggests no obvious way of finding a generating set meeting its bound.

Web link: www.math.uni-bielefeld.de/~baumeist/wop2017/: click on Vortragsfolien and choose Gareth Tracey. Read about Frobenius 20 here: manu.amiot.free.fr/pdf/Articles/chords.pdf.

Further reading: *Permutation Groups* by P.J. Cameron, CUP, 1999.

