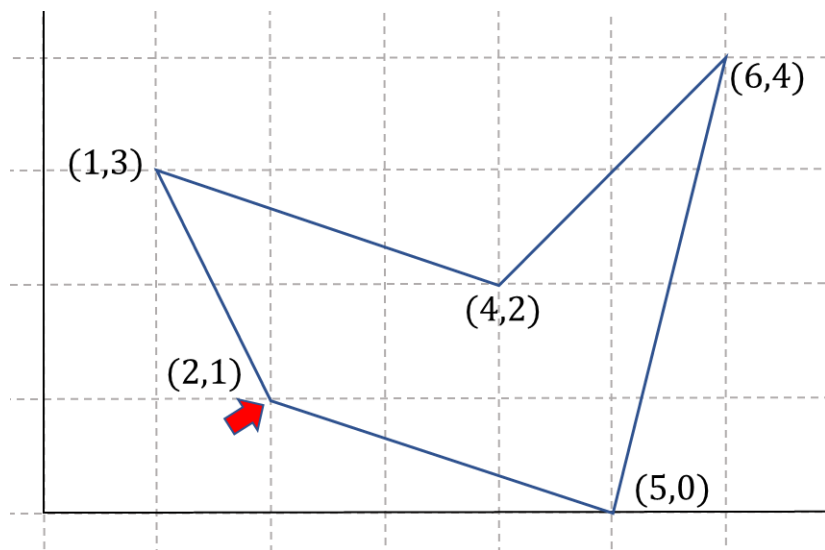




# THEOREM OF THE DAY

**The Shoelace Formula** Suppose the  $n$  vertices of a simple polygon in the Euclidean plane are listed in counterclockwise order as  $(x_0, y_0), \dots, (x_{n-1}, y_{n-1})$ . Then the area  $A$  of the polygon may be calculated as:

$$A = \frac{1}{2} (x_0 y_1 - x_1 y_0 + \dots + x_{n-2} y_{n-1} - x_{n-1} y_{n-2} + x_{n-1} y_0 - x_0 y_{n-1}).$$



## Example

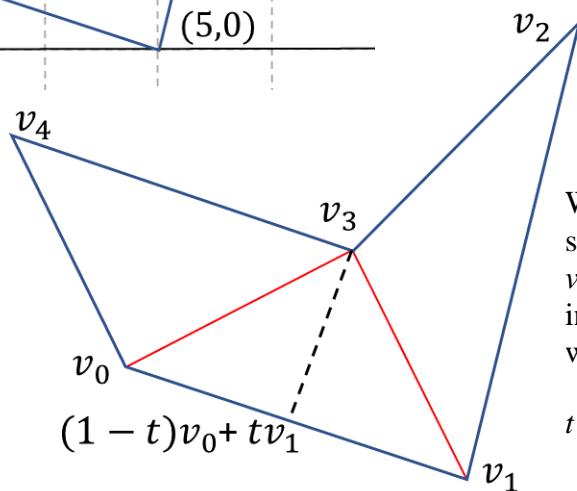
Because the polygon on the left has lattice point vertices, Pick's theorem gives its area as:

$I + B/2 - 1 = 6 + 6/2 - 1 = 8$ , where  $I$  (resp.  $B$ ) is number of interior (resp. boundary) lattice points. We can confirm that the Shoelace formula gives the same value, calculating counterclockwise from the arrow:

$$\begin{aligned} \frac{1}{2} \times ( & 2 \times 0 - 1 \times 5 \\ & + 5 \times 4 - 0 \times 6 \\ & + 6 \times 2 - 4 \times 4 \\ & + 4 \times 3 - 2 \times 1 \\ & + 1 \times 1 - 3 \times 2 ) = 8. \end{aligned}$$

## An Application

We may triangulate a polygon on  $n$  vertices by adding  $n - 3$  diagonals, as illustrated on the right. We would like to test if some straight line joining a triangle vertex to the opposite polygon edge bisects the area of the polygon. In our diagram this requires a value of  $t \in [0, 1]$  for which the polygons  $v_0, (1-t)v_0 + tv_1, v_3, v_4$  and  $(1-t)v_0 + tv_1, v_1, v_2, v_3$  have equal area.



The Shoelace formula was invented in 1769 by Albrecht Meister, but it is widely attributed to Gauss who made significant discoveries about polygons at the age of 18 in the 1790s. It may now be seen as an application of Green's Theorem (1828).

We apply the Shoelace formula, simplifying via  $v_i \wedge v_i = 0$  and  $v_i \wedge v_j = -v_j \wedge v_i$ , just as for the invariants  $e_i$ . We get an equation which we may solve for  $t$ , giving:

$$t = \frac{v_0 v_1 + v_1 v_2 + v_2 v_3 + v_3 v_0 - (v_0 v_3 + v_3 v_4 + v_4 v_0)}{2(v_0 v_1 + v_1 v_3 + v_3 v_0)}, \text{ omitting the } \wedge \text{ s for greater clarity.}$$

And we recognise three applications of the Shoelace formula! Denote by  $A_{cl}$  the polygon area clockwise from of our chosen triangle; by  $A_{co}$  the remaining polygon area; and by  $A_{\Delta}$  the area of the triangle itself. Then

$$t = \frac{A_{co} - A_{cl}}{2A_{\Delta}}.$$

For our example polygon, again using Pick,  $A_{cl} = 5/2$ ,  $A_{co} = 8 - 5/2 = 11/2$  and  $A_{\Delta} = 5/2$ . This gives  $t = (11/2 - 5/2)/5 = 3/5$  (a little to the right of our dotted line).

## Using the exterior algebra

- Invariants  $e_1, \dots, e_n$ , multiplied and added over a field (e.g.  $\mathbb{R}$ ) to give 'formal' expressions.

$$\text{E.g. } E = e_1 e_2 e_3 - 3 e_1 e_3 e_2 + \sqrt{2} e_3^3.$$

- Expressions multiply using the exterior (wedge) product  $E \wedge F$  (omitted for single invariants as in the above example) using the following rule:

$$e_i e_j = \begin{cases} 0 & i = j \\ -e_j e_i & i \neq j \end{cases}$$

Thus our example expression above simplifies:

$$E = e_1 e_2 e_3 + 3 e_1 e_2 e_3 + 0 = 4 e_1 e_2 e_3.$$

- Encoding  $v_i = (x_i, y_i)$  as  $x_i e_1 + y_i e_2$ , we have  $v_i \wedge v_j = (x_i y_j - x_j y_i) e_1 e_2$ .
- Now we can express the Shoelace formula very concisely:

$$A = \frac{1}{2} (v_0 \wedge v_1 + \dots + v_{n-1} \wedge v_0)$$

(or, more precisely, the coefficient of  $e_1 e_2$  in this summation).

Web link: [www.math.tolaso.com.gr/?p=1451](http://www.math.tolaso.com.gr/?p=1451)

Further reading: *The Shoelace Book* by Burkard Polster, AMS, 2006.