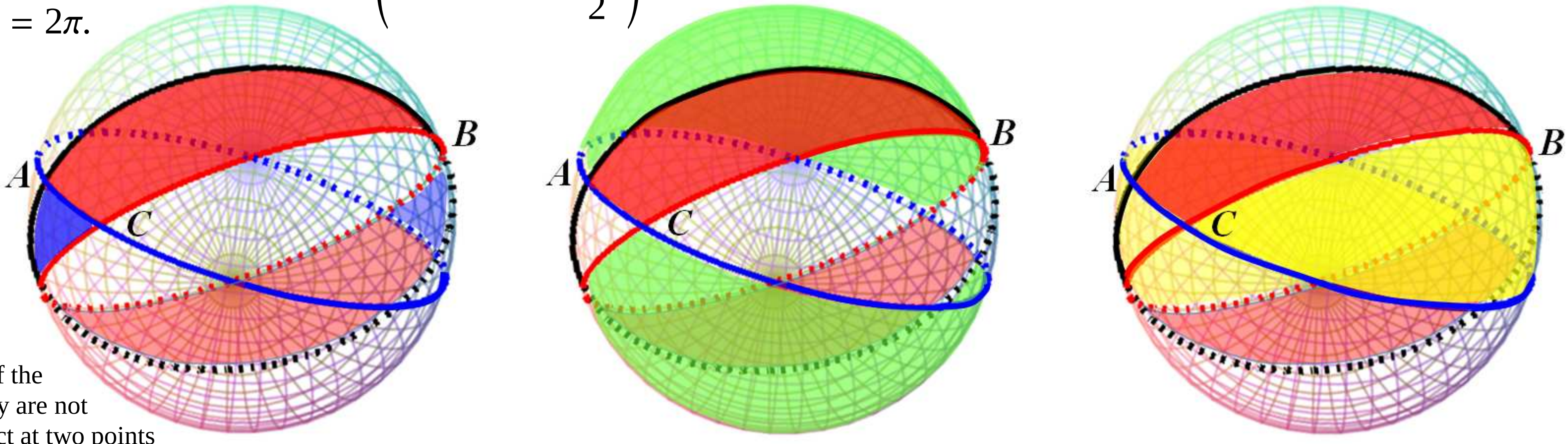
THEOREM OF THE DAY

Girard's Theorem *A spherical triangle on the surface of a sphere of radius r, with angles A, B and C, has area, T, given by*

$$T = r^2\left(A + B + C - \frac{1}{2}\tau\right),$$

where $\tau = 2\pi$.



(a) (b) (c)

Two great circles (centres and radii coinciding with those of the sphere), provided they are not the same one, intersect at two points on the surface of the sphere. These points are antipodal and they describe a crescent on the surface which is called a *lune*. It stretches exactly half-way around the sphere, from pole to pole (not necessarily the North and South poles). A third great circle will cut this crescent into two spherical triangles. In the figure above, the triangle ABC has been created by the intersection of a great circle through A and B, another through C and B, and a third through A and C. The AB and CB circles form the lune shown in the figure at (a); the AC circle cuts this into the red triangle ABC and the blue 'bottom' triangle. If we follow the AB and BC great circles round the 'back' of the sphere their intersection creates a second lune identical in shape and area to the first. Circle AC cuts this lune into two triangles as well, shown in paler colours at (a), and these triangles are congruent to the first pair but located antipodally.

Now we extend our lunar dissection: our three intersecting great circles give us three antipodal pairs of lunes, all passing through, and duplicating, triangle ABC. Our original pair of lunes, shown at (a), has the lunar angle B. At (b), the lunar angle is taken to be C and the original triangle ABC is now paired with one (green) stretching over the top of the sphere. And at (c), the lunar angle is A and the second (yellow) triangle extends to the right.

A sphere of radius r has surface area is $2\tau r^2$. If two great circles meet in a *lunar* angle θ, $0 < \theta \leq \tau$, then the proportion of surface area which is occupied by the lune they create is θ/τ. So we have

$$\text{area of lune with lunar angle } \theta = \frac{\theta}{\tau} \times 2\tau r^2 = 2r^2\theta.$$

Denote by L_A, L_B and L_C the areas of the three lunes with angles A, B and C, respectively. Recall that these areas each have an antipodal duplicate. Denote by T the area of triangle ABC; this also has its antipodal duplicate.

Add up all the duplicate pairs of lunar areas: you get the whole sphere but with T counted three times and its antipodal duplicate likewise counted three times:

$$\begin{aligned} 2L_A + 2L_B + 2L_C &= 2r^2\tau + 4T \\ \text{so } T &= \frac{1}{2}\left(L_A + L_B + L_C - r^2\tau\right) \\ &= \frac{1}{2}\left(2r^2A + 2r^2B + 2r^2C - r^2\tau\right) \\ &= r^2\left(A + B + C - \frac{1}{2}\tau\right), \end{aligned}$$

which is Girard's Theorem.

This theorem was published by Albert Girard in 1626 but has also been attributed to Thomas Harriot, 1603.

Web link: www.princeton.edu/~rvdb/WebGL/GirardThmProof.html

Further reading: *Heavenly Mathematics: The Forgotten Art of Spherical Trigonometry* by Glen Van Brummelen, Princeton University Press, 2012, chapter 7.