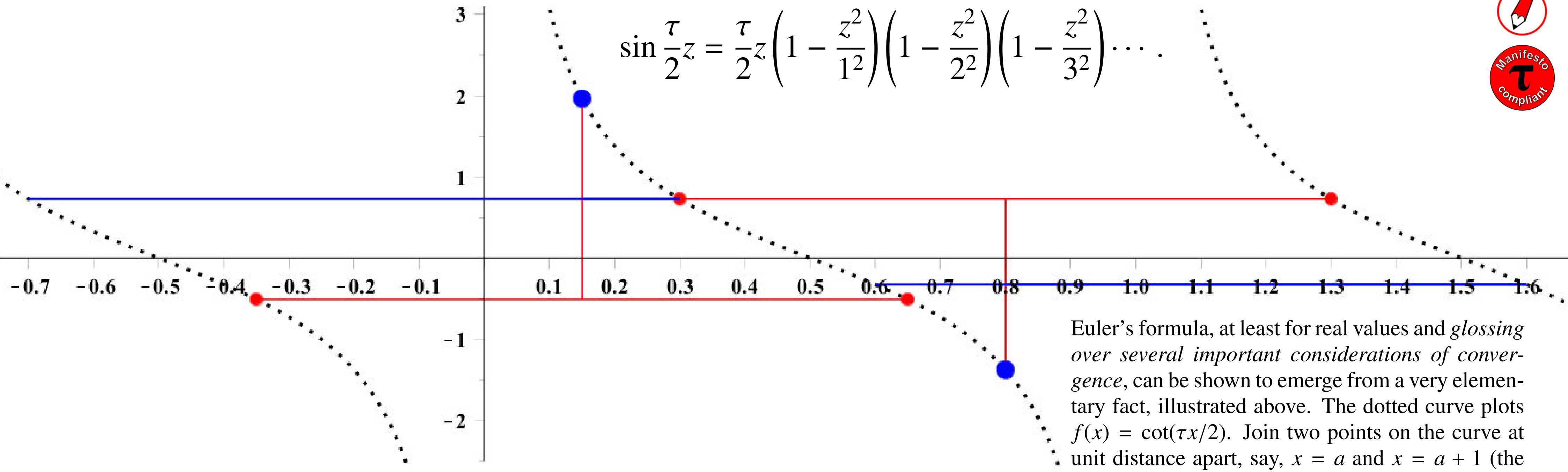
THEOREM OF THE DAY

Euler's Product Formula for Sine *For any complex number z,*

$$\sin\frac{\tau}{2}z = \frac{\tau}{2}z\left(1-\frac{z^2}{1^2}\right)\left(1-\frac{z^2}{2^2}\right)\left(1-\frac{z^2}{3^2}\right)\cdots.$$



Euler's formula, at least for real values and *glossing over several important considerations of convergence*, can be shown to emerge from a very elementary fact, illustrated above. The dotted curve plots $f(x) = \cot(\tau x/2)$. Join two points on the curve at unit distance apart, say, $x = a$ and $x = a + 1$ (the small red dots), with a horizontal line segment. From its midpoint, draw a vertical line to meet the curve (the larger blue dots). Then the bisector of this vertical line meets the curve at $2a$ and $2a + 1$.

Translated into algebra this is a half-angle formula: $f(x) = \frac{1}{2}\left(f(x/2) + f((x \pm 1)/2)\right)$. For example, $f(0.6)$ is the mean of $f(0.3)$ and $f(0.8)$, using the '+1' option for the second term, which is equally the value of $f(1.6)$ using the '−1' option (the right-hand construction above). Apply the same formula to $f(0.3)$: $f(0.3) = \frac{1}{2}(f(0.15) + f(0.65)) = \frac{1}{2}(f(0.15) + f(-0.35))$ (the left-hand construction). This suggests a repeated substitution into the half-angle formula: for the $\pm$ term we choose sign according to the rule $(x \pm k)/2^n \longrightarrow \frac{1}{2}((x \pm k)/2^n \mp 1)$, for $1 \le k \le 2^{n-1} - 1$, while for $k = 0$ or 2^{n-1} we just take the '+1' option We arrive, after n steps, at $f(x) = 2^{-n}\left(f(x/2^n) + f\left((x + 2^{n-1})/2^n\right) + \sum_{k=1}^{2^{n-1}-1} f((x \pm k)/2^n)\right)$. Now by L'Hospital's Rule, $\lim_{x\to 0} x\cot x = 1$ so, for non-integer α, and denoting $\tau/2$ by π, $\frac{\pi\alpha}{2^n} f\left(\frac{\alpha}{2^n}\right) \to 1$, as $n \to \infty$, or $\frac{1}{2^n} f\left(\frac{\alpha}{2^n}\right) \to \frac{1}{\pi\alpha}$. Letting n approach infinity in our expression for $f(x)$ we derive **Euler's Partial Fraction Cotangent Expansion**: *for non-integer x,* $f(x) = \frac{1}{\pi x} + \sum_{k=1}^{\infty} \frac{1}{\pi(x \pm k)}$. Rearranging gives, for $x \notin \mathbb{Z}$, $\frac{\tau}{2}\cot\frac{\tau}{2}x = \frac{1}{x} + \sum_{k=1}^{\infty}\frac{2x}{x^2 - k^2} = \frac{1}{x} + \sum_{k=1}^{\infty}\frac{-2x/k^2}{1 - x^2/k^2}$. The latter rearrangement allows us to arrive directly at Euler's sine formula by integrating both sides: $\ln\sin\frac{\tau}{2}x = \ln x + \sum_{k=1}^{\infty}\ln\left(1 - x^2/k^2\right) + \ln C = \ln\left(Cx\prod_{k=1}^{\infty}\left(1 - x^2/k^2\right)\right)$. Exponentiating: $\sin\frac{\tau}{2}x = Cx\prod_{k=1}^{\infty}\left(1 - x^2/k^2\right)$. And invoking L'Hospital again for $\lim_{x\to 0}\sin\frac{\tau}{2}x/x = \frac{\tau}{2}$, we find that $C = \tau/2$ and Euler's formula follows.

Euler's sine product first appears in §16 of his E41 (Eneström Index) and the cotangent expansion in E61 (again §16). Made rigorous over the complex numbers they are, respectively, applications of Weierstrass's Factorisation Theorem and Mittag-Leffler's Theorem (both 1876).

Web link: people.math.sfu.ca/~tarchi/turnermsc2007.pdf (700KB). Euler originals via Euler Archive.
Further reading: *Analysis by its History* by Ernst Hairer and Gerhard Wanner, Springer, 1996.