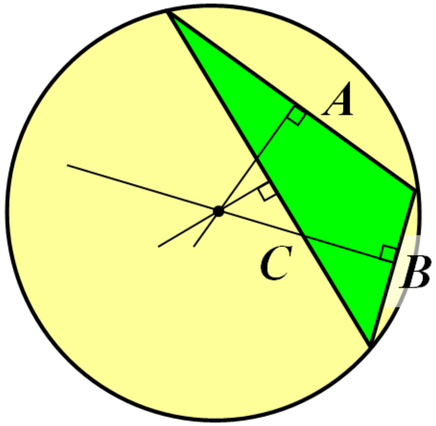
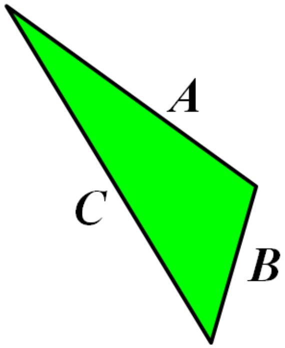
THEOREM OF THE DAY

Brahmagupta's Formula *The area K of a cyclic quadrilateral with side lengths a, b, c, d and semiperimeter $s = (a + b + c + d)/2$ is given by*

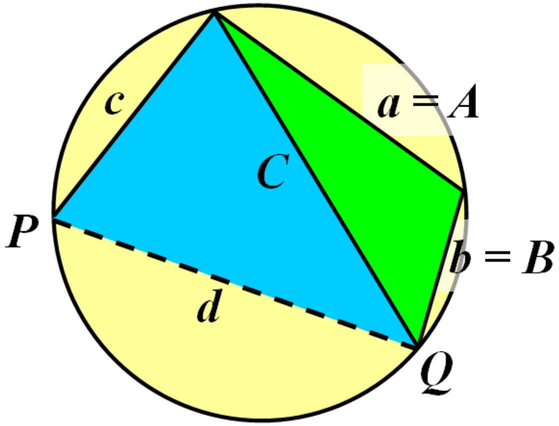
$$K = \sqrt{(s-a)(s-b)(s-c)(s-d)}.$$



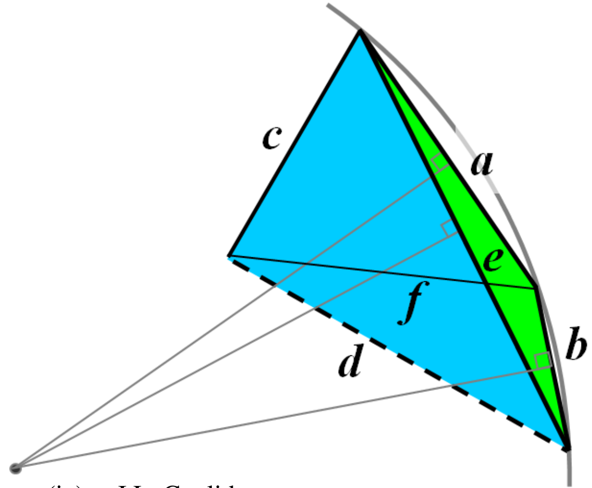
(i) Euclid of Alexandria
Elements, Book IV, (*c.* 300 BC)
Circumcircling the triangle



(ii) Heron of Alexandria
Metrica, (1st century AD)
Sides → area (triangle)



(iii) Brahmagupta
The Brahmasphutasiddhanta, (628 AD)
Sides → area (cyclic quadrilateral)



(iv) J.L. Coolidge
Amer. Math. Monthly, (1939)
Sides → area (any quadrilateral)

(i) The familiar formula for triangular area (1/2 × base × height) was known to Greek mathematicians for at least three hundred years before Euclid catalogued all of known geometry in his *Elements*, including (Book IV, Proposition 5) the construction of the unique *circumcircle* which passes through the vertices of a triangle, by intersecting the perpedicular bisectors of its sides.

(ii) Three hundred years after *that* came the famous **Heron's formula**: *the area K of a triangle with sides A, B, C and semiperimeter $s = (A + B + C)/2$ is given by $K = \sqrt{(s-A)(s-B)(s-C)s}$.*

(iii) Although Greek mathematics was apparently unknown to medieval Indian (as opposed to Islamic) scholars, Brahmagupta effectively put Heron's triangle back into the circle: take two non-overlapping circumscribed triangles sharing a common edge (C in the picture); the result is a *cyclic quadrilateral*, one whose vertices all lie on a circle. And now the area of the quadrilateral replaces the final s in Heron's formula by $s - d$. If point P is allowed to approach point Q then d becomes zero and c becomes C, recovering Heron.

(iv) Another thirteen hundred years pass and the circumscribing circle is removed once more in American mathematician Julian Lowell Coolidge's **quadrilateral area formula:** *the area of an arbitrary convex quadrilateral with sides a, b, c, d, a opposite to d, with diagonals e, f, and with semiperimeter s, is given by $K = \sqrt{(s-a)(s-b)(s-c)(s-d) - \frac{1}{4}(ad + bc + ef)(ad + bc - ef)}$.* This generalises Brahmagupta by virtue of another classic of antiquity, **Ptolemy's Theorem:** *quadrilateral a, b, c, d, a opposite to d, with diagonals e, f, is cyclic if and only if $ad + bc = ef$.*

Brahmagupta (598–670) was the first mathematician to treat zero as a number in its own right.

Web link: www-history.mcs.st-and.ac.uk/Biographies/Brahmagupta.html.
Further reading: *Journey Through Genius* by William Dunham, John Wiley & Sons, 1990.