



# THEOREM OF THE DAY

**The Analyst's Travelling Salesman Theorem** Define a dyadic cube,  $Q \subset \mathbb{R}^n$ , to be a Cartesian product  $Q = \prod_{i=1}^n [m_i 2^{-k}, (m_i+1) 2^{-k}]$ , of closed intervals of length  $l(Q) = 2^{-k}$ ,  $m_i, k \in \mathbb{Z}, k \geq 0$ . Let  $3Q$  denote the cube having the same centroid as  $Q$  and having side length  $3l(Q)$ , and for  $K$  a bounded subset of  $\mathbb{R}^n$ , let  $r_K(Q)$  denote the minimum radius of any cylinder enclosing  $K \cap 3Q$ . Now set  $\beta_K(Q) = r_K(Q)/l(Q)$ . Then  $K$  is contained in some rectifiable curve if and only if

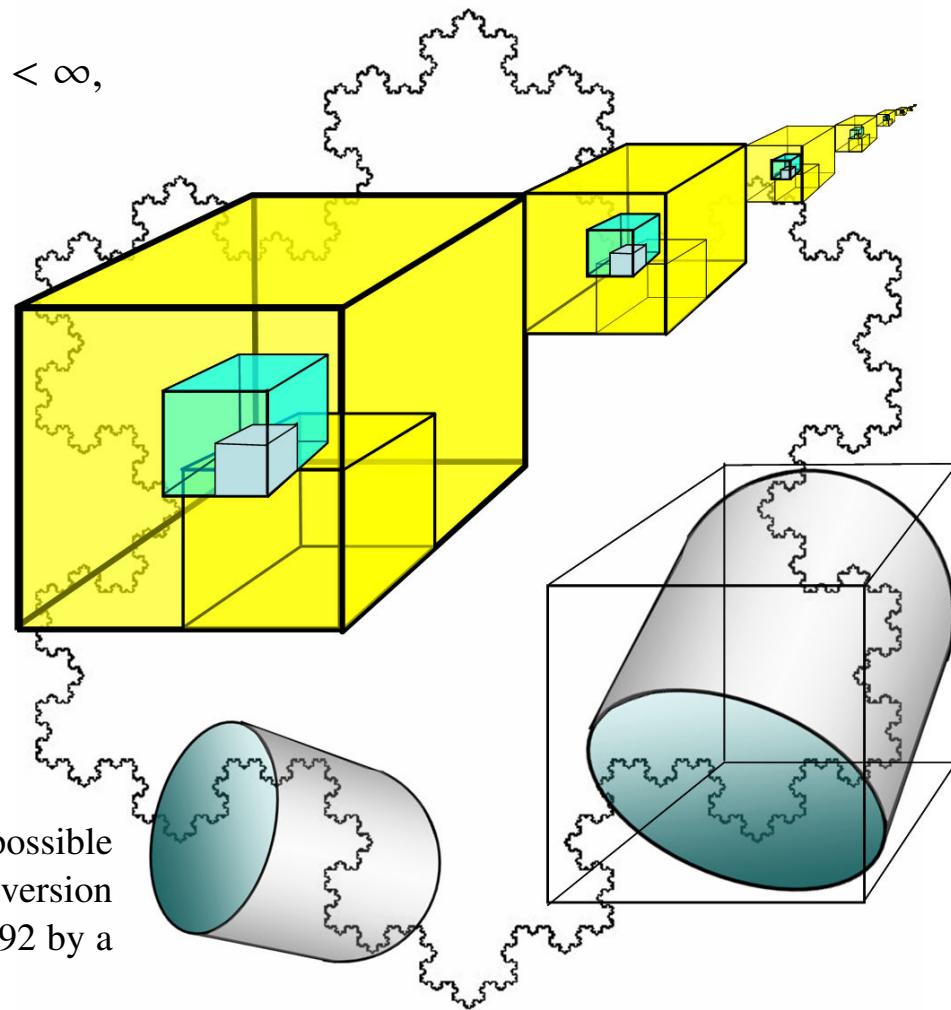
$$\sum_Q \beta_K^2(Q) l(Q) < \infty,$$

where the sum is over all dyadic cubes.

The operation of the theorem is shown schematically on the right. Every point in  $n$ -dimensional space lies in an infinite sequence of dyadic cubes, each cube  $Q$  having side-length,  $l(Q)$ , half that of the preceding cube in the sequence. Now expand each cube to  $3Q$  and consider its intersection with some bounded set  $K$ . For each  $Q$ , the value  $\beta_K(Q) = r_K(Q)/l(Q)$  is a measure of how far  $K$  deviates from a straight line, in the vicinity of  $Q$  and scaled by  $l(Q)$ . Choose, by angling as convenient, the thinnest cylinder which will contain this deviation. The principle of the theorem is that, for a 'well-behaved' set of points,  $K$ , decreasing dyadic cubes will quickly enclose  $K$  within an infinitesimally thin cylinder, allowing the sum in the theorem to converge.

The set  $K$  shown on the right is the famous 'Koch snowflake' which is bounded and continuous but not rectifiable (because it has infinite length). Even within an infinitesimally small  $3Q$ , the snowflake exhibits the same complex structure as seen here — the dyadic cubes cannot tame it and the summation over all of them will be infinite.

The traditional 'travelling salesman' must tour as economically as possible a finite set of cities. The publication of this continuous ('analyst's') version in  $\mathbb{R}^2$  by Peter Jones of Yale University in 1990 was followed in 1992 by a proof for arbitrary  $n$  by Kate Okikiolu, then at Princeton.



**Web link:** [arxiv.org/abs/cs/0512042](https://arxiv.org/abs/cs/0512042) proves a 'computable' version of the theorem and provides an excellent account of the mathematical background. The Koch snowflake image is from [www.scientificweb.com/testreport/mathbench4/](http://www.scientificweb.com/testreport/mathbench4/).

**Further reading:** *Geometric Measure Theory: a Beginner's Guide* by Frank Morgan, Academic Press, 2000.

