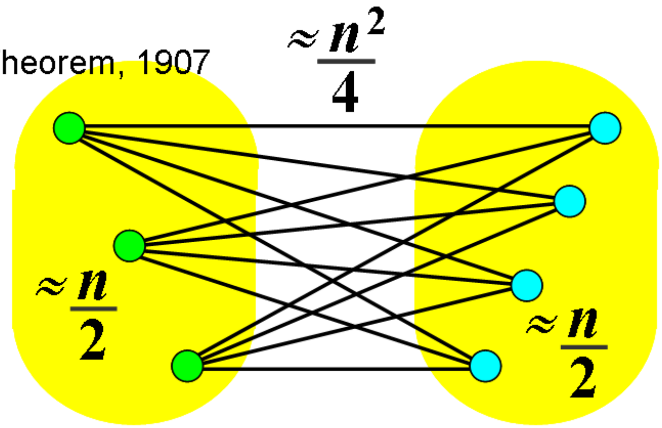
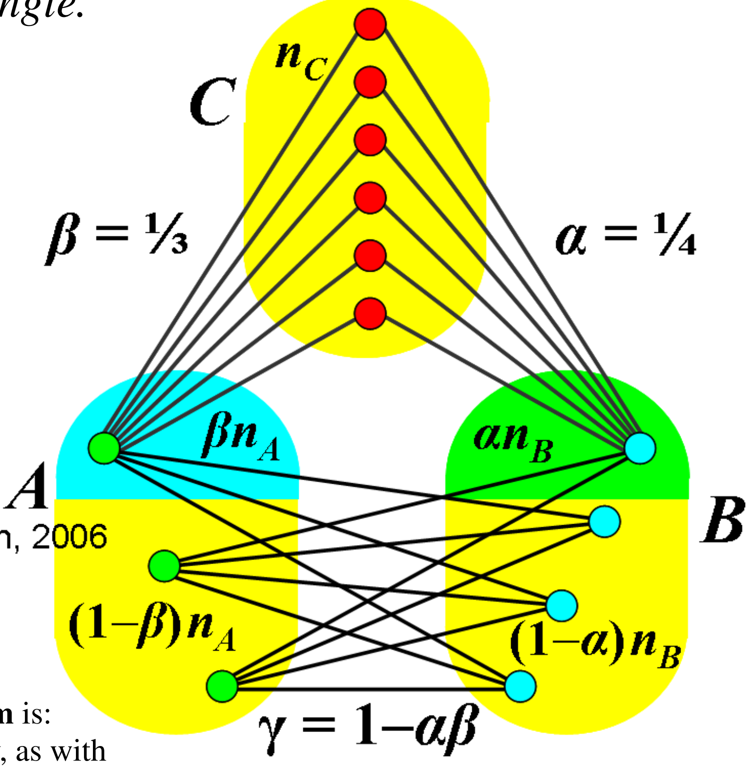
THEOREM OF THE DAY

A Tripartite Turán Theorem *Let G be a tripartite graph with parts A, B and C. Let $d(A,B)$ denote the density of edges between A and B, i.e. $d(A,B) = (\text{no. of edges between } A \text{ and } B)/(|A||B|)$, and similarly for $d(A,C)$ and $d(B,C)$. Now denote $d(A,B)$, $d(B,C)$ and $d(A,C)$ by γ, α and β, respectively, and let φ denote the golden ratio. Then, if $\alpha, \beta, \gamma > 1/\varphi$, G has a triangle.*

Mantel's Theorem, 1907





Bondy, Shen, Thomassé and Thomassen, 2006

In 1907 the Dutch mathematician Willem Mantel published the solution to a problem posed by him in *Wiskundige Opgaven* (Mathematical Exercises): if a graph G on n vertices has m edges then how large must m be for G necessarily to contain a triangle (a cyclic path of three edges, otherwise known as K_3)? The answer, now known as **Mantel's Theorem** is: $m > \lfloor n^2/4 \rfloor$, *and this is best possible*, because a bipartite graph contains no triangles but may, as with the one shown above left, have exactly $\lfloor n^2/4 \rfloor$ edges. The critical edge density, then, is 1/2: if more than 1/2 of the $\binom{n}{2}$ possible edges are present then a triangle is inevitable. In the bipartite graph all the edge density occurs between the two parts of the partition; it is natural to ask, what are the edge densities, α, β and γ between the three parts of a tripartite graph that will force a triangle? Adrian Bondy, Jian Shen, Stéphan Thomassé and Carsten Thomassen published the answer in 2006: a triangle is forced when:

$$\alpha\beta + \gamma > 1, \text{ and } \beta\gamma + \alpha > 1, \text{ and } \gamma\alpha + \beta > 1. \qquad (1)$$

Now the famous equation $\varphi^2 - \varphi - 1 = 0$ rearranges to give $(1/\varphi)^2 + 1/\varphi = 1$, so that $1/\varphi$ supplies a simultaneous critical density for α, β and γ. Again, this is best possible: in the tripartite graph above right, the n_A vertices of part A are subdivided in proportion to β, the edge density from C; and all this density is incident with the 'top' βn_A vertices of A. B is similarly subdivided in proportion to α. There is no density between the top vertices of A and B which makes triangles impossible, and we have $\alpha\beta + \gamma = \alpha\beta + (1 - \alpha\beta) = 1$, so one of the inqualities in (1) fails to hold.

Although extremal graph theorists trace their subject back to Mantel's famous problem it is the 1941 generalisation from triangles K_3 to arbitrary complete graphs K_r by Paul Turán that underlies modern work in the area.

Web link: www.cimpa-icpam.org/IMG/pdf/BandungLlado.pdf

Further reading: *Graph Theory* by J.A. Bondy and U.S.R. Murty, Springer, 2008, Chapter 12.

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