THEOREM OF THE DAY

The Piff-Welsh Theorem *If M is a transversal matroid and F is a field then M is representable over some finite extension of F. In particular, M is representable over fields of any characteristic.*

A collection of subsets of E={a, b, c, d, e}

A={a, c, d, e} B={a, b, d, e} C={b, c}

for which {b, c, d} is a transversal:

{a, **c**, d, e} {a, b, **d**, e} {**b**, c}

but {a, d, e} is not

{**a**, c, d, e} {a, b, **d**, e} {b, c}
?

its incidence matrix

	a	b	c	d	e
A	1		1	1	1
B	1	1		1	1
C		1	1		

and a transversal matroid representation

	a	b	c	d	e
A	1	0	1	1	1
B	1	1	0	2	-2
C	0	1	1	0	0

Adding a new set membership

a	b	c	d	d'			d+2d'
1	0	1	1	0	→		1
1	1	0	0	1	→		2
0	1	1	0	0	→		0
x	x		p 1	q -1	a,b,d	a,b,d'	
x		x	1	-1	a,c,d	a,c,d'	
	x	x	1	1	b,c,d	b,c,d'	
			α	β	→	1, 2	

Suppose $\mathcal{A}$ is a collection of subsets of some set E. A partial transversal is a subset of E whose elements each 'chose' a distinct member of $\mathcal{A}$ to which they belong. These partial transversals are the independent sets of a matroid on E; Piff and Welsh's theorem allows us to rephrase this as: the elements of E can be represented by vectors in such a way that a subset of E is a partial transversal if and only if the corresponding set of vectors is linearly independent (cannot produce the zero vector as a non-trivial weighted sum).

Suppose E is to be represented by vectors over the field $\mathbb{Q}$ of rationals. It may be that the incidence matrix, as illustrated above centre, is a representation; the vector for $x \in E$ has 1's for the sets in $\mathcal{A}$ containing x and 0's elsewhere. In our example, this will not do: $\{a, d\}$ is a partial transversal but the incidence vectors of a and d are linearly dependent (their difference is the zero vector). Replacing the second 1 in d's vector with -1 is not enough: now the transversal $\{b, c, d\}$ has linearly dependent vectors; this is confirmed by calculating the determinant of the corresponding 3×3 matrix — it is zero.

A rational representation that *is* valid is shown above right; its construction is illustrated bottom left (with E limited, for the sake of clarity, to a, b, c and d). Suppose we have a representation for the simpler case in which d is absent from set B. We add d to B while extending the representation: first, calculate the determinants of each pair of square submatrices containing either the current d vector or the new membership vector d'. E.g., the submatrix comprising columns a, b, d has determinant $p = 1$, while the determinant for columns a, b, d' is $q = -1$. Next, choose α and β so that every determinant pair (p, q) has $(\alpha p, \beta q) \neq (0, 0)$; this is always possible, provided we are calculating over a large enough field (hence the need for field extensions in the theorem). Finally, replace d and d' with $\alpha d + \beta d'$.

The axioms which define matroids were specified in the 1930s as a generalisation of linear independence of vectors. The extent to which a class of matroids can still be described by linear algebra, i.e. the question of the representability of the matroids over different fields, is a major component in understanding their behaviour. This theorem, proved by Dominic Welsh and his DPhil student Michael Piff in 1970, completely resolves the question for the important case of transversal matroids.

Web link: www.ams.org/featurecolumn/archive/matroids1.html

Further reading: *Matroid Theory, 2nd Edition* by James G. Oxley, Oxford University Press, 2011, chapter 11.