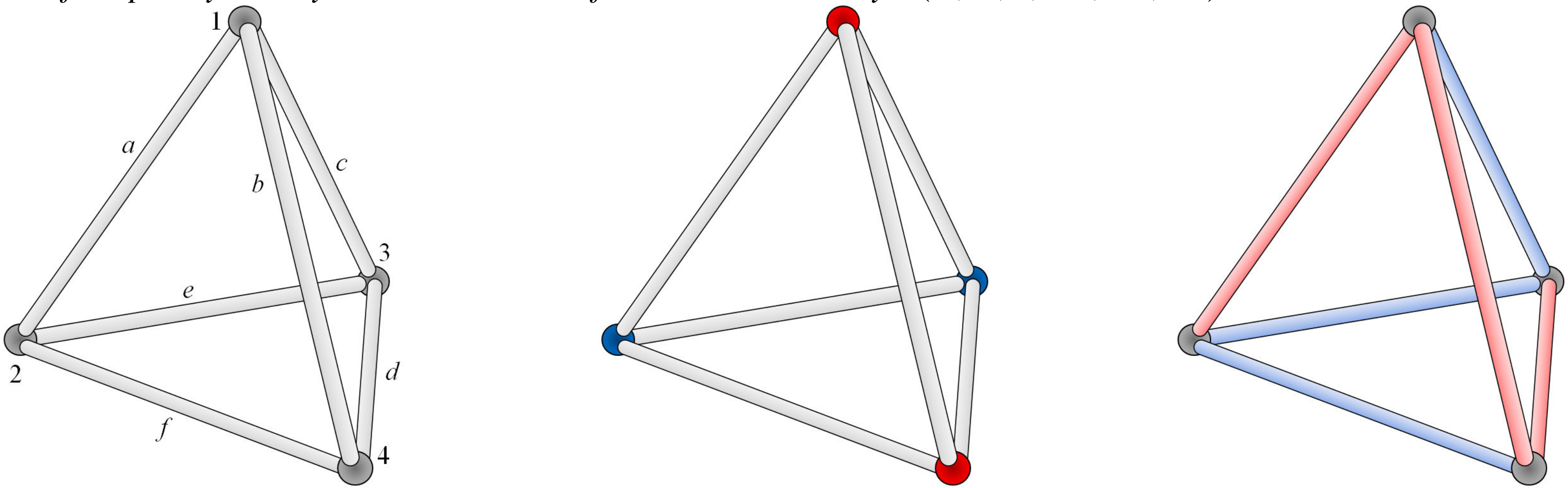
THEOREM OF THE DAY

The Pólya–Redfield Enumeration Theorem *Denote by $Z(G, \Omega; s_1, \ldots, s_t)$ the cycle index of a permutation group G acting on a set Ω. For a set of distinct invariants $L = \{x_1, \ldots, x_n\}$, denote by $x^{(k)}$, $k = 1, \ldots, t$, the formal sum $x_1^k + \ldots + x_n^k$. Then the number of different ways in which Ω may be labelled with elements of L up to symmetry under the action of G is enumerated by $Z(G, \Omega; x, x^{(2)}, \ldots, x^{(t)})$.*



We construct the cycle index for Ω_V, the set of vertices of the regular tetrahedron, and for Ω_E, its set of edges. The group acting on these sets will be the rotational symmetries of the tetrahedron, which is the alternating group A_4. As an abstract group, A_4 may be defined as multiplication of the twelve even permutations of $\{1, \ldots, 4\}$:

(1)(2)(3)(4), (1 2)(3 4), (1 3)(2 4), (1 4)(2 3), (1 2 3)(4), (1 3 2)(4), (1 2 4)(3), (1 4 2)(3), (1 3 4)(2), (1 4 3)(2), (1)(2 3 4), (1)(2 4 3),

('even' by virtue of their numbers of disjoint cycles having the same parity as the permuted set, given that 1-cycles are listed, as above). If the vertices are labelled $1, \ldots, 4$, as above left, then our even permutations directly represent the rotational symmetries of the tetrahedron: for example, (1 2)(3 4) is a half-rotation about an axis through the midpoints of edges a and d. The cycle index encodes these symmetries in terms of their cycle lengths: s_i records cycles of length i, hence s_1^4 records a permutation of four 1-cycles (i.e. (1)(2)(3)(4)); $3s_2^2$ records three pairs of 2-cycles; $8s_1s_3$ records eight permutations with a 1-cycle and a 3-cycle. And finally $Z(A_4, \Omega_V; s_1, s_2, s_3)$ averages these over the whole group to give $\frac{1}{12}(s_1^4 + 3s_2^2 + 8s_1s_3)$. Now we may enumerate, say, in how many ways the vertices may be coloured red, r, and blue, b, up to symmetry: $Z(A_4, \Omega_V; r + b, r^2 + b^2, r^3 + b^3) = b^4 + b^3r + b^2r^2 + br^3 + r^4$. The coefficient of b^ir^j counts colourings with i blue vertices and j red ones: we see that the colouring of the tetrahedron above centre is unique, up to symmetry. When the tetrahedral symmetries act on edges instead of vertices the group is still A_4 but the disjoint cycle structures have changed and this is reflected in a different cycle index (see above left again; this time 1-cycles are omitted for conciseness):

$$1,\ (a\ d)(b\ e),\ (a\ d)(c\ f),\ (b\ e)(c\ f),\ (a\ b\ c)(d\ e\ f),\ (a\ b\ f)(c\ d\ e),\ (a\ c\ b)(d\ f\ e),\ (a\ c\ e)(b\ d\ f),\ (a\ e\ c)(b\ f\ d),\ (a\ f\ b)(c\ e\ d),\ (a\ f\ e)(b\ d\ c) \rightarrow \frac{1}{12}(s_1^6 + 3s_1^2s_2^2 + 8s_3^2).$$

This time the calculation reveals that the half-red-half-blue colouring, above right, is one of four possibilities up to symmetry.

The 1937 rediscovery by George Pólya of this neglected 1927 result of J. Howard Redfield turned it into a classic of combinatorics.

Web link: crypto.stanford.edu/pbc/notes/polya/; the Redfield story: match.pmf.kg.ac.rs/content46.htm, the article by E. Keith Lloyd.
Further reading: *Combinatorics: Ancient and Modern* by Robin J. Wilson and John J. Watkins, OUP, 2013, Chapter 12.