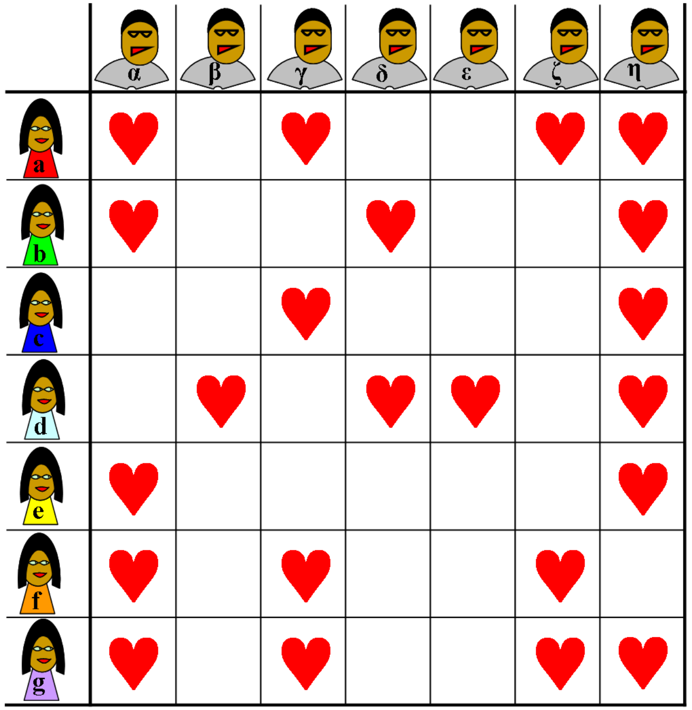
THEOREM OF THE DAY

The Marriage Theorem *In a set of n women each specifies a list of the men she is prepared to marry, as a subset, W_i, of a set of n men, $i = 1, \ldots, n$. Assuming that any man will accept any offer of marriage then there is a monogamous espousal matching woman i with a man from W_i, $i = 1, \ldots, n$, if and only every subset X of women like a combined total of at least $|X|$ men.*



Hall's Marriage Theorem extends, more generally, to a theorem about finding a *transversal* for a (possibly uncountable) collection of finite subsets of a set: a representative from each subset with no two representatives the same.

The Frobenius–Kőnig Theorem *An $n \times n$ (0-1)-matrix contains a $n \times n$ permutation matrix among its non-zero entries if and only if no $r \times s$ submatrix of zeros has $r + s > n$.*

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 1 \end{pmatrix}$$

A *permutation matrix* has a single 1 in each row and column, all other entries being zero. An obvious obstacle to this is when there is an $n \times 1$ zero submatrix, occupying one complete row. Certainly then $r + s = n + 1 > n$. It is not obvious that extending this to zero submatrices of all sizes covers *every* obstacle to extracting a permutation matrix. Here, and in the marriage attempt opposite, a 5×3 submatrix spoils things.

Two theorems for the price of one! And, in fact, these superficially different results, both minimizing an obstacle to achieve a maximum result, belong to a collection of essentially equivalent *minimax* theorems which were discovered independently, in this instance by Philip Hall in 1935, and by Georg Frobenius (1917) and Dénes Kőnig (1931), respectively.

Web link: robertborgersen.info/Presentations/GS-05R-1.pdf

Further reading: *Combinatorial Optimization: Algorithms and Complexity* by C. Papadimitriou and K. Steiglitz, Dover Publications, 2000.