

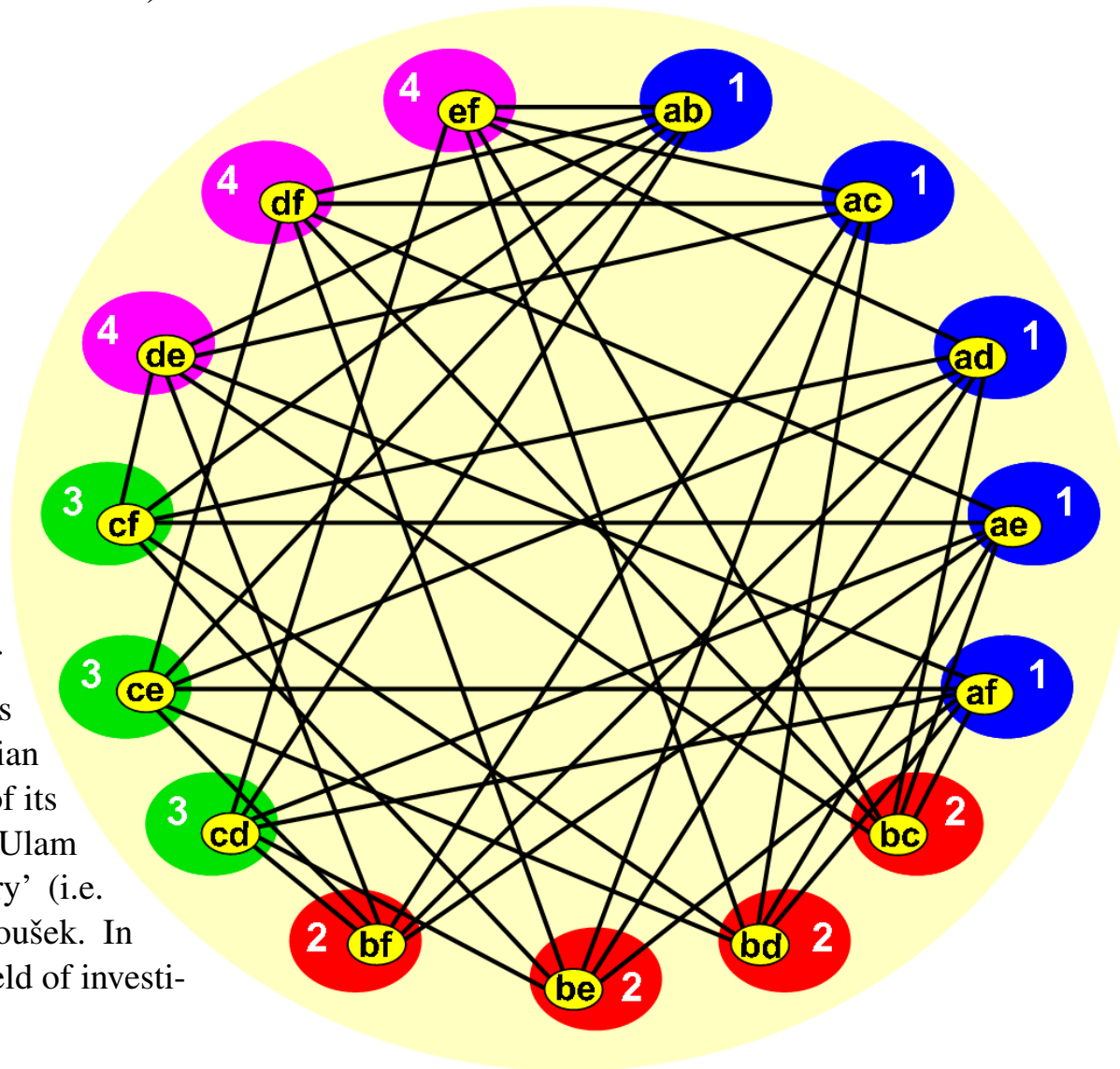


# THEOREM OF THE DAY

**Kneser's Conjecture** Let  $C_{n,k}$ ,  $1 \leq 2k - 1 \leq n$ , denote the set of all  $k$ -element subsets of  $\{1, \dots, n\}$ . Let  $C_1 \cup \dots \cup C_t = C_{n,k}$ ,  $1 \leq t$ , be a partition of  $C_{n,k}$  such that any two sets in any of the  $C_i$  intersect nontrivially (i.e.,  $c, c' \in C_i \Rightarrow c \cap c' \neq \emptyset$ , for  $1 \leq i \leq t$ ). Then  $n - 2k + 2 \leq t$ .

Denote by  $[n]$  the set  $\{1, \dots, n\}$ . The property of  $[n]$  asserted by this theorem is viewed in a natural way in terms of graph colouring. Define the *Kneser graph*  $KG_{n,k}$  (pron. K-nay-zer) by taking the  $\binom{n}{k}$  subsets of  $[n]$  of size  $k$  as vertices, joining two by an edge precisely when they are disjoint. The graph  $KG_{6,2}$  is shown on the right, with  $[6]$  represented by the letters 'a', ..., 'f'. Now, a  $t$  colouring of  $KG_{n,k}$  partitions the vertices into  $t$  colour classes so that no edge joins vertices of the same colour class. The theorem says that the smallest value of  $t$ , that is, the *chromatic number* of  $KG_{n,k}$ , is  $n - 2k + 2$ . When  $n = 6$  and  $k = 2$ , this gives a value of 4, and the 4-colouring of  $KG_{6,2}$  shown as large numbered ovals on the right can readily be seen to extend to an  $(n - 2k + 2)$ -colouring of  $KG_{n,k}$  in the general case. An  $(n - 2k + 1)$ -colouring, however, is never possible.

Martin Kneser (1928–2004) proposed this property of set systems in 1955, in connection with a study of quadratic forms. Apart from its inherent interest, its eventual proof, in 1978 by the Hungarian mathematician László Lovász, sparked enormous interest because of its reliance on a deep theorem of topology, the Borsuk-Ulam theorem. Not until 2000 did a difficult but 'elementary' (i.e. purely combinatorial) proof appear, due to Jiří Matoušek. In the meantime, Lovász's work had opened up a new field of investigation: topological combinatorics.



**Web link:** [www.emis.de/newsletter/current/](http://www.emis.de/newsletter/current/) (pages 16–19)

**Further reading:** *Using the Borsuk-Ulam Theorem: Lectures on Topological Methods in Combinatorics and Geometry* by Jiří Matoušek, Springer, Berlin, 2003.

