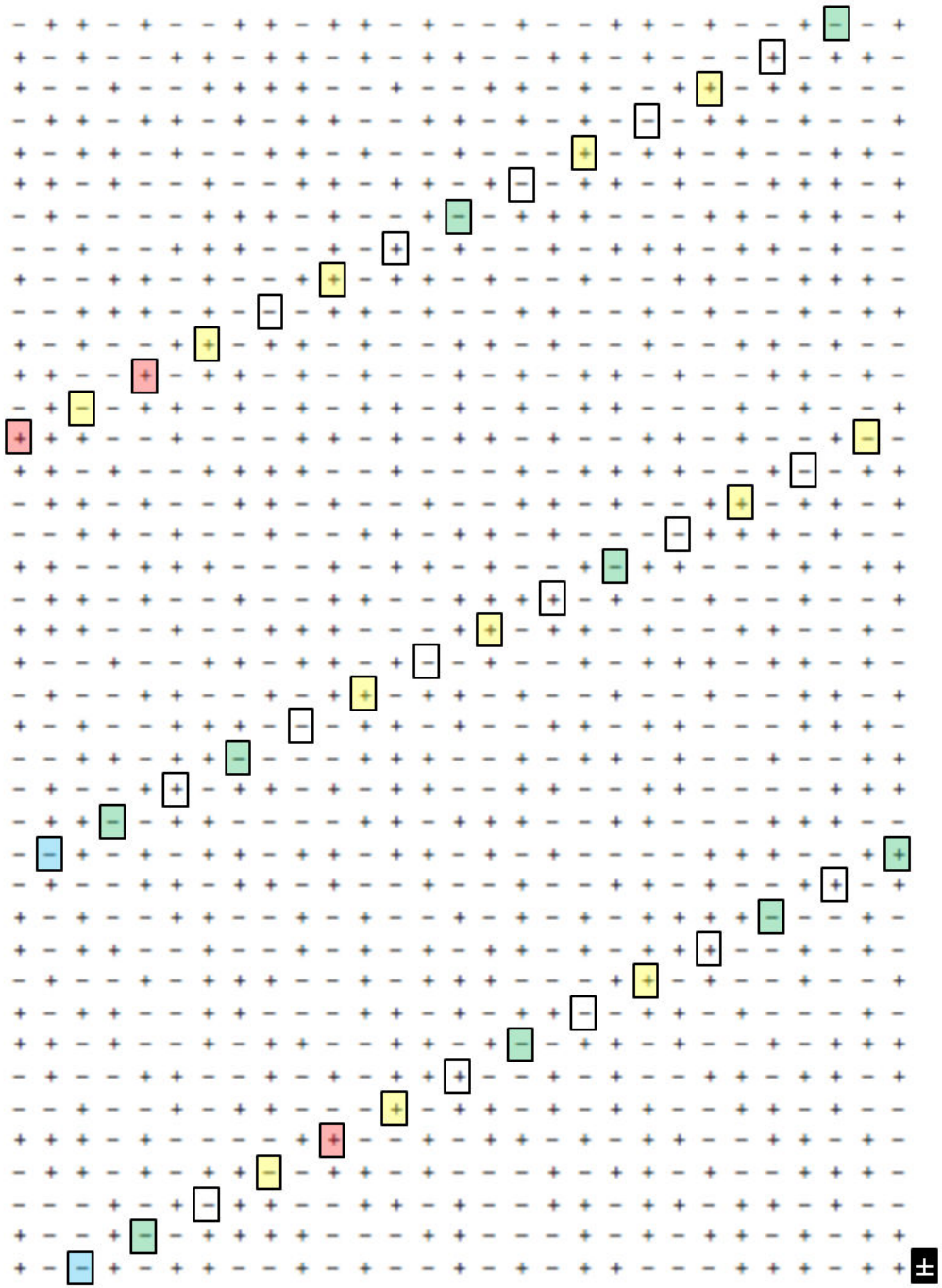
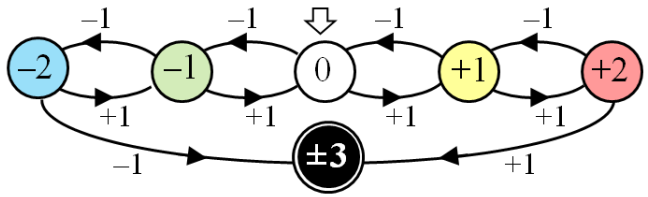
THEOREM OF THE DAY



The Erdős Discrepancy Problem *Let C be a positive integer and let $(x_i)_{i\geq 1}$ be an infinite sequence of values from $\{-1, 1\}$. Then there exist positive integers d, n, for which the sum $x_d + x_{2d} + \ldots + x_{(n-1)d} + x_{nd}$ exceeds C in absolute value.*



This theorem asserts that, given any infinite ±1 sequence, some homogeneous arithmetic progression (i.e. with initial term equal to its common difference) will index a subsequence whose terms eventually sum to $\pm(C+1)$. We say that the *discrepancy* of the sequence exceeds C. This is illustrated on the left for $C = 2$: the sequence, arranged row by row, contains 1160 terms ('+' and '−' denote 1 and −1, respectively). The sum of all the terms is 2: this means that if the sequence were to continue with a +1 then its first 1161 terms would sum to +3. But now observe that the arithmetic progression with common difference $d = 27$ indexes terms summing to −2: these terms are highlighted with colour-coded boxes, with the colour indicating the evolving sum. If the sequence were extended with a −1 then the first $1161/27 = 43$ terms of our arithmetic progression would index terms in the sequence which summed to −3.

The evolution of subsequence summations indexed by an arithmetic progression may be tracked by the finite automaton shown above. Starting in the central zero state we move left for a −1 and right for a +1. A −1 in the left-most state, or a +1 in the right-most, will move to the stop state, recording a discrepancy exceeding ±2. The behaviour of this automaton has an ingenious representation as an expression in propositional logic: an assignment of truth values to the variables of this expression which make the whole expression true (a *satisfying assignment*) corresponds precisely to a finite ±1 sequence whose discrepancy does not exceed 2. The sequence on the left was produced in this way.

Erdős' discrepancy question dates from the early 1930s but was independently raised by the number theorist Nikolai Chudakov in 1956. It was answered in the affirmative for all positive C by Terence Tao in 2015 ($C = 1$ was settled in 1993 by Adrian Mathias and $C = 2$ by Boris Konev and Alexei Lisitsa in 2014 using the logic-based analysis illustrated here).

Web link: gilkalai.wordpress.com/2015/10/22/edp-reflections-and-celebrations/ Erdős' first published reference to his conjecture appears as Problem 9 in 1957, no. 13 at renyi.hu/~p_erdos/Erdos.html. The ±1 sequence on the left is reproduced from Konev & Lisitsa's paper: arxiv.org/abs/1402.2184.

Further reading: *Combinatorics, Geometry and Probability: A Tribute to Paul Erdős*, Béla Bollobás and Andrew Thomason (eds.), Cambridge University Press, paperback edition, 2004.