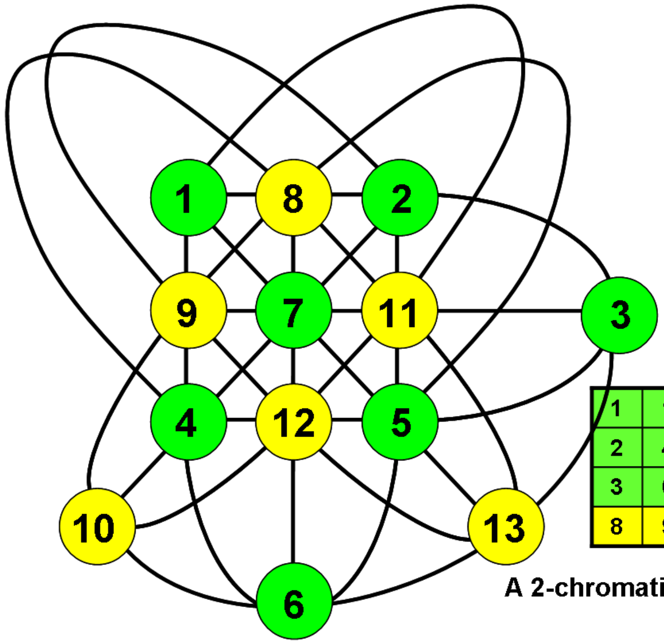
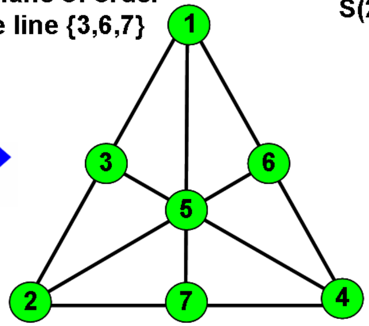
THEOREM OF THE DAY

The Design of the Century *There exists a 2-chromatic Steiner system $S(2,4,100)$ in which every block contains three points of one colour and one point of the other colour.*



Projective plane of order 2, minus the line {3,6,7}



1	1	1	1	2	2	2	3	3	3	4	5	6
2	4	5	10	4	5	9	4	6	7	8	8	7
3	6	7	11	7	6	12	5	10	9	11	9	8
8	9	13	12	10	11	13	12	13	11	13	10	12

A 2-chromatic *S*(2,4,13), aka the projective plane of order 3

A 2-chromatic S(2,4,100) (detail)

0	12	0	16	0	9	0	5	28	29
1	20	4	7	6	14	14	23	1	39
9	10	33	11	24	43	21	∞	14	2
44	44	45	45	46	46	47	47	48	48
4	36	40	1	8	1	4	2	4	37
6	5	3	2	19	7	35	7	22	5
23	7	6	21	40	29	41	18	26	19
44	44	45	45	46	46	47	47	48	48
8	2	8	9	4	5	1	22	0	3
11	3	24	13	31	26	10	34	38	11
13	38	5	42	3	2	31	3	∞	35
44	44	45	45	46	46	47	47	48	48

Cycle the yellow points in steps of 4, modulo 44 (except for ∞, which is fixed) and the blue points in steps of 5 modulo 99, offset by 44. Thus, the 8th quadruple of the top row above cycles after 11 steps:
$\{5, 23, \infty, 47\} \to \{9, 27, \infty, 52\} \to \cdots \{41, 15, \infty, 92\} \to \{1, 19, \infty, 97\}$

A Steiner system $S(t, k, v)$ is an experimental design in which v treatments (points) are subjected to a number of trials (blocks or lines), with each trial involving k treatments and with each and every collection of t treatments appearing together in precisely one trial. The projective plane of order 3 is an $S(2,4,13)$ system and is shown, above left, to be *2-chromatic*: that is, its 13 points have been coloured yellow and green so that every line involves points of both colours. If we take just the lines having three green points, and extract these green triples then, centre top, we *nearly* get an $S(2,3,7)$; one line is missing: $\{3,6,7\}$. In fact, for our green points to constitute a so-called *Steiner triple system* in their own right, we would need the two-colouring specified in the theorem: every line or block must have three points of one colour and one of the other. This is impossible for $S(2,4,13)$; in fact, it can be shown to be impossible for all v, $4 < v < 100$. When $v = 100$, it is achieved by an $S(2,4,100)$ system which is partially displayed on the right with a yellow-blue colouring. Only 30 of its 3-yellow-1-blue blocks are shown; the others are recovered by cycling as specified underneath, the final collection yielding a yellow $S(2,3,45)$ with 330 blocks. The 3-blue-1-yellow blocks, 495 in number, are specified in terms of a ‘base set’ of 45 (details via the web link below). The completed design has 825 blocks.

The ‘Design of the Century’ remained a tantilisingly elusive animal for over 25 years until design theory and computing power were sufficiently advanced for the Open University team of Forbes, Grannell and Griggs to be able to produce a specimen. With the target 825 blocks represented by just 75, the search still took over 40 weeks of personal computer time.

Web link: math.stackexchange.com/questions/136372/: Ed Pegg’s contribution has a link to Forbes et al’s original paper.

Further reading: *Combinatorial Designs and Tournaments* by Ian Anderson, Clarendon Press, 1997.